Optimal multiple change-point detection and Localization

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https://arxiv.org/abs/2010.11470
https://arxiv.org/abs/2011.07818

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Offline Change-point Analysis

First Aim

In the mean change-point univariate setting :

1. understand the **information-theoretical** scalings.
2. set up **specifications** for change-point procedures
3. exhibit several procedures achieving them.

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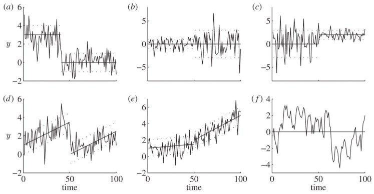

Beaulieu et al.('12)

Second Aim

Start a **roadmap** towards more general models with e.g. **sparse multivariate change point** models.

(Sub)-Gaussian univariate mean change-point Model

Data : Time series $\mathbf{Y} \in \mathbb{R}^n$

$$y_i = \theta_i + \epsilon_i, \qquad \text{where } \epsilon_i \overset{ind.}{\sim} \mathcal{SG}(1) \ ,$$

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Notation : change-point vector $\boldsymbol{\tau}^*$
$1 < \tau_1^* < \ldots < \tau_K^* \leq n$
s.t. $\boldsymbol{\theta}$ is constant over $[\tau_k^*, \tau_{k+1}^*)$.

Height $\Delta_k = \theta_{\tau_k^*} - \theta_{\tau_k^*-1}$

Radius $r_k = \frac{(\tau_{k+1}^* - \tau_k^*)(\tau_k^* - \tau_{k-1}^*)}{\tau_{k+1}^* - \tau_{k-1}^*}$

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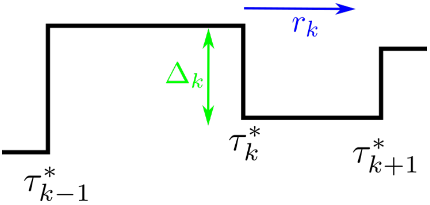

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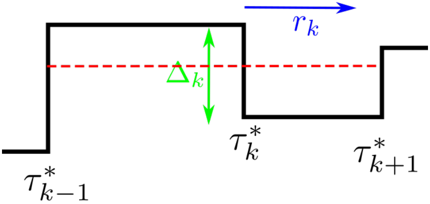

Definition of the Energy of τ_k^*

The **Square Energy** of τ_k^* is $\qquad E_k^2 = r_k \Delta_k^2$

l_2 distance between $\boldsymbol{\theta}$ and best approximation by a piece-wise constant vector on $\boldsymbol{\tau}^{(-k)} = (\tau_1^*, \ldots, \tau_{k-1}^*, \tau_{k+1}^*, \ldots)$.

Two mathematical perspectives on change-point detection

- Denoising/Estimation : Estimating $\boldsymbol{\theta} \rightsquigarrow$ small risk $\mathbb{E}[\|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_2^2]$
- Clustering/Segmentation : Recover the change-points $\boldsymbol{\tau}^*$.

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Denoising perspective :
Minimax-Optimal rates (for $K \geq 2$) $\qquad K\left[1 + \log\left(\frac{n}{K}\right)\right]$

achieved e.g. by penalized least-squares [Birgé and Massart, 2001, Gao et al., 2020]

Quadratic computational complexity by dynamic programming...

Change-point detection as a clustering problem

Several lines of literature :

- At Most One Change-point (AMOC) $[K \leq 1]$. **Least-square** estimator detects $\widehat{K} = 1$ if $E_1 \gg \sqrt{\log\log(n)}$ and $|\widehat{\tau}_1 - \tau_1^*| = O(\Delta_1^{-2})$ [Csorgo and Horváth, 1997]. V.et al.('20) $\rightsquigarrow$ detection iif $E_1 \geq \sqrt{2\log\log(\frac{n}{r_1})}$.

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- Greedy or Aggregation methods
  Binary segmentation [Scott and Knott, 1974] = iterative bisection.
  Many recent variants
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computational complexity $O(n\log(n))$.

Typical Results in the literature

Theorem (Typical modern result. *sloppy version* ; [Wang et al., 2020, Fryzlewicz, 2018, Kovács et al., 2020])

If $\min_k E_k^2 \gtrsim \log(n)$, *then whp* $\widehat{K} = K$ *and*

$$d_H(\widehat{\boldsymbol{\tau}}, \boldsymbol{\tau}^*) = \max_{k=1,\dots,K} |\widehat{\tau}_k - \tau_k^*| \lesssim \frac{\log(n)}{\min_k \Delta_k^2}$$

But see [Frick et al., 2014] and [Cho and Kirch, 2019] for tighter results in different senses.

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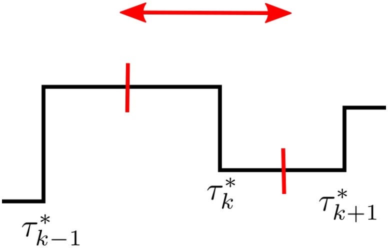
Questions :

- Is $\min_k E_k^2 \gtrsim \log(n)$ really necessary ?
- What if a few change-points have a small energy ?
- Is the second $\log(n)$ necessary ?

Two sub-problems

Change-Point Detection
= Detecting the existence of the change-point



Objectives

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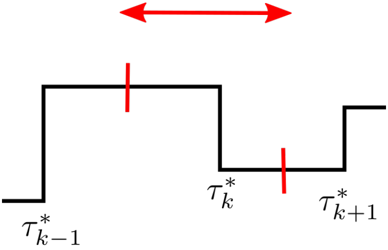
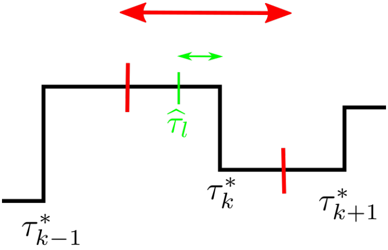


Change-Point Localization
= small estimation error
$d_{H,1}(\widehat{\boldsymbol{\tau}}, \tau^*_k) = \min_l |\widehat{\tau}_l - \tau^*_k|$



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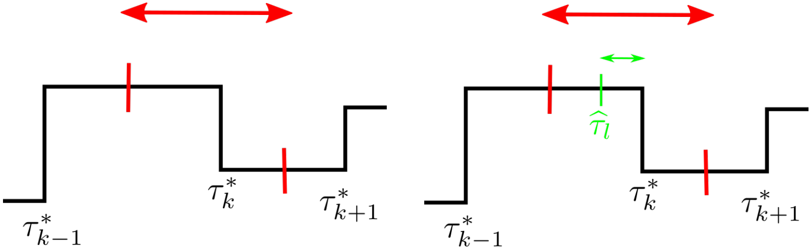
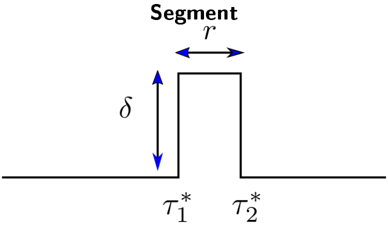


Questions

- What is the energy requirement for detection ?
- How is the transition between detection and localization ?
- Is penalized least-square optimal ? For which penalty ?

Simpler problem : testing $\boldsymbol{\theta} = 0$ versus

$$\boldsymbol{\theta} \in \Theta[r, \delta] = \left\{ \boldsymbol{\theta} \in \mathbb{R}^n : \exists \tau \text{ such that } \theta_i = \delta \mathbb{1}_{i \in [\tau, \tau + r)} \right\} .$$



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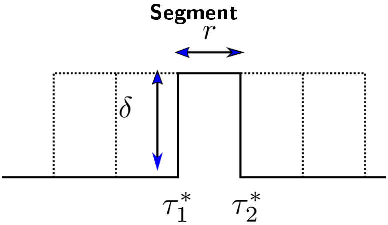

$\lfloor \frac{n}{2r} \rfloor$ possible positions

For each τ, sufficient statistic
$Z_\tau = r^{-1/2} \sum_{i=\tau}^{\tau+r-1} y_i \sim \begin{cases} \mathcal{N}(0,1) \\ \mathcal{N}(r^{1/2}\delta, 1) \end{cases}$

Gaussian Change-point Detection

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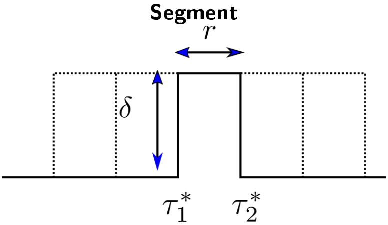

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If $r \ll n$, then $r = r_1(1 + o(1)) = r_2(1 + o(1))$.

Proposition (Segment Detection ≈ [Arias-Castro et al., 2011])

If $\delta\sqrt{r} \le \sqrt{2(1 - o(1)) \log[n/(2r)]}$, *then testing* **better than random guess** *is impossible,*

$$\inf_T \mathbb{P}_0[T = 1] + \sup_{\boldsymbol{\theta} \in \Theta[\delta, r]} \mathbb{P}_{\boldsymbol{\theta}}[T = 0] \ge 1 - o(1) .$$

High-energy Change-points

$\kappa > 1$; $q > 0$.

Definition

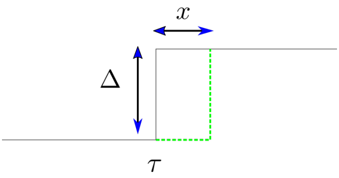
τ_k^* is a (κ, q)-**high-energy** change-point if $\mathbf{E}_k(\boldsymbol{\theta}) > \kappa\sqrt{2\log\left(\frac{n}{r_k}\right) + q}$.

Remarks :

- For small r_k, then $\log(n/r_k) \asymp \log(n)$.
- For small $r_k \asymp n$, then $\log(n/r_k) \asymp 1$
- The additive term q will play the role of a global probability.

Simplified setting :

- one change-point with known means $\boldsymbol{\mu} = (\mu_1, \mu_2)$
- Two possible positions for τ^* : τ or $\tau + x$.



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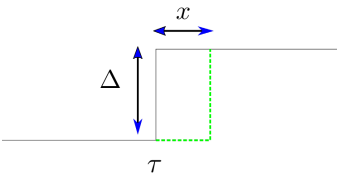

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Write $\Delta = \mu_2 - \mu_1$. *For any* $r \geq 2$,

$$\inf_{\widehat{\tau}} \sup_{\tau^* \in \{2,\dots,n\}} \mathbb{P}_{\boldsymbol{\theta}(\tau^*, \boldsymbol{\mu})} \left(|\widehat{\tau} - \tau^*| \geq \frac{r}{\Delta^2} \right) \gtrsim e^{-cr} \quad ,$$

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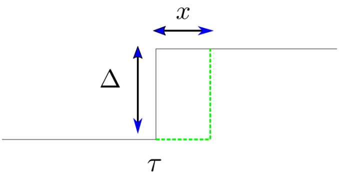

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Small Δ : At best, $|\hat{\tau} - \tau^*| \asymp \Delta^{-2}$ and has a sub-exponential tail.

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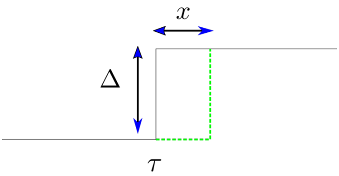

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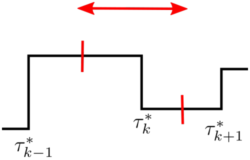
Large Δ : At best, $\widehat{\tau} = \tau^*$ with proba higher than $1 - c' e^{-c\Delta^2}$.

Desiderata for a suitable change-point procedure

Under an event $\mathcal{A}$ of high (*to be discussed*) probability, then $\widetilde{\boldsymbol{\tau}}$

(NoSp). No spurious change-point is detected :

$$\begin{cases} \left|\{\widetilde{\boldsymbol{\tau}}\} \cap \left(\frac{\tau^*_{k-1}+\tau^*_k}{2}, \frac{\tau^*_k+\tau^*_{k+1}}{2}\right]\right| \leq 1 \ , \ \text{for all } k \text{ in } \{2,\ldots,K-1\} \ ; \\ \left|\{\widetilde{\boldsymbol{\tau}}\} \cap \left[2, \frac{\tau^*_1+\tau^*_2}{2}\right]\right| \leq 1 \ ; \ \left|\{\widetilde{\boldsymbol{\tau}}\} \cap \left(\frac{\tau^*_{K-1}+\tau^*_K}{2}, n\right]\right| \leq 1 \ . \end{cases}$$



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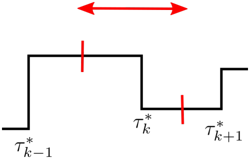
For all k in $[K]$, if τ^*_k is a (κ, q)-high-energy change-point then

$$d_{H,1}(\widetilde{\boldsymbol{\tau}}, \tau^*_k) \leq \min\left\{\frac{\tau^*_{k+1} - \tau^*_k}{2}, \frac{\tau^*_k - \tau^*_{k-1}}{2}, c\frac{\log\left(1 \vee n\Delta_k^2\right) + q}{\Delta_k^2}\right\} \;\; .$$



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(Loc). High-energy change-points are localized at the optimal rate.
Any high-energy change-point τ_k^* satisfies

$$\mathbb{P}\left(d_{H,1}(\widetilde{\boldsymbol{\tau}}, \tau_k^*)\mathbb{1}_{\mathcal{A}} \geq c\frac{x}{\Delta_k^2}\right) \lesssim e^{-x}, \quad \forall x \geq 1 \ .$$

Penalized least-square estimator

$\boldsymbol{\tau}$= vector of tentative change-points
$\boldsymbol{\Pi}_{\boldsymbol{\tau}}$ = projector onto the space of piece-wise constant vectors with changes at $\boldsymbol{\tau}$

$$\widehat{\tau} = \arg\min_{\tau} \mathrm{Cr}_0(\mathbf{Y}, \boldsymbol{\tau}) = \arg\min_{\boldsymbol{\tau}} \|\mathbf{Y} - \boldsymbol{\Pi}_{\boldsymbol{\tau}} \mathbf{Y}\|^2 + L \; \mathrm{pen}_0(\boldsymbol{\tau}, q) \quad ,$$

BIC Penalty $\quad \mathrm{pen}_{BIC}(\boldsymbol{\tau}, q) = 2|\boldsymbol{\tau}| \log(n)$

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Remarks :

- Additive Penalty $\rightsquigarrow$ dynamic programming (and its refinements [Killick et al., 2012])
- Over-penalizes **small** segments.
- Differs from complexity penalties $\mathrm{pen}_{BM}(\boldsymbol{\tau}, q) = (|\boldsymbol{\tau}| + 1)(1 + \log(n/|\boldsymbol{\tau}|))$.

Connection between CUSUM and Least-square penalty

Definition (CUSUM Statistic)

For $\mathbf{t} = (t_1, t_2, t_3)$, $\mathbf{C}(\mathbf{Y}, \mathbf{t}) = \left[\overline{\mathbf{Y}}_{[t_2, t_3)} - \overline{\mathbf{Y}}_{[t_1, t_2)}\right] \sqrt{\frac{(t_2 - t_1)(t_3 - t_2)}{t_3 - t_1}}$

Connection between CUSUM and Least-square penalty

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Lemma (deletion of a change-point e.g. [Wang et al., 2020])

$\boldsymbol{\tau}^{(-l)} = (\tau_1, \ldots, \tau_{l-1}, \tau_{l+1}, \ldots)$

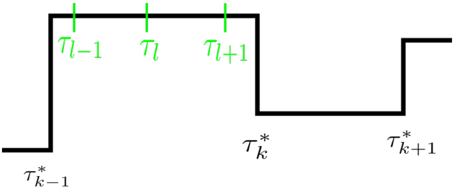
$$\|\mathbf{Y} - \mathbf{\Pi}_{\boldsymbol{\tau}} \mathbf{Y}\|^2 - \|\mathbf{Y} - \mathbf{\Pi}_{\boldsymbol{\tau}^{(-l)}} \mathbf{Y}\|^2 = -\mathbf{C}^2[\mathbf{Y}, (\tau_{l-1}, \tau_l, \tau_{l+1})] \ .$$

$$\begin{aligned} \mathrm{Cr}_0(\mathbf{Y}, \boldsymbol{\tau}) - \mathrm{Cr}_0(\mathbf{Y}, \boldsymbol{\tau}^{(-l)}) \quad &= \quad -\mathbf{C}^2\big(\mathbf{Y}, (\tau_{l-1}, \tau_l, \tau_{l+1})\big) \\ &\quad +L\left[2\log\left(\frac{n(\tau_{l+1} - \tau_{l-1})}{(\tau_{l+1} - \tau_l)(\tau_l - \tau_{l-1})}\right) + q\right] \ . \end{aligned}$$

Local Optimality and uniform Control of the CUSUM

Consider $\boldsymbol{\tau}$ such that
$\boldsymbol{\theta}$ is constant on $[\tau_{l-1}, \tau_{l+1})$

Goal : show that $\boldsymbol{\tau} \neq \widehat{\boldsymbol{\tau}}$?



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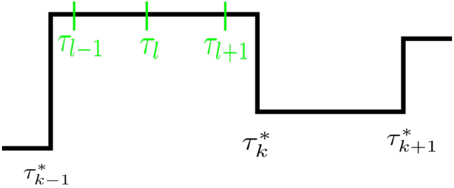

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$\boldsymbol{\tau} \neq \widehat{\boldsymbol{\tau}}$ as long as $\mathbf{C}^2\big(\boldsymbol{\epsilon}, (\tau_{l-1}, \tau_l, \tau_{l+1})\big)$ small enough.

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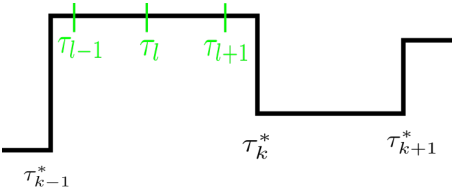

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Local Optimality $\rightsquigarrow$ Uniform bound for the CUSUM

Lemma (Multi-scale chaining ; in the spirit of [Dumbgen and Spokoiny, 2001])

$$\mathcal{A}_q = \left\{ |\mathbf{C}(\boldsymbol{\epsilon}, \mathbf{t})| \leq 2\sqrt{2\log\left(\frac{n(t_3 - t_1)}{(t_3 - t_2)(t_2 - t_1)}\right) + q}, \quad \forall \mathbf{t} = (t_1, t_2, t_3) \right\} .$$

We have $\mathbb{P}[\mathcal{A}_q] \geq 1 - ce^{-c'q}$.

First Analysis of Penalized Least-square

High Energy condition : $E_k(\theta) \gtrsim \sqrt{\log(\frac{n}{r_k}) + q}$

Proposition (V. et al. ('20))

For any L and q large enough, under $\mathcal{A}_q$, the penalized least-square estimator $\widehat{\tau}$ satisfies

(a) **(NoSp)** *No Spurious Jump is detected.*

(b) **(Detec)** *All high-energy change-points τ_k^* are detected*

$$d_{H,1}(\widehat{\boldsymbol{\tau}}, \tau_k^*) \leq \min \left\{ \frac{\tau_{k+1}^* - \tau_k^*}{2}, \frac{\tau_k^* - \tau_{k-1}^*}{2}, \kappa_L \frac{\log\left(n\Delta_k^2\right) + q}{\Delta_k^2} \right\}$$

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Remarks :

- Allow arbitrarily low-energy jumps.
- Local condition for high energy $\rightsquigarrow \log(n/r_k)$ (see also [Frick et al., 2014, Chan and Chen, 2017])
- Dependency in q is optimal with respect to the probability $1 - ce^{-c'q}$
- Complexity-based penalties are highly suboptimal.

Localization **(Loc)** by Penalized Least-squares

Proposition (V. et al. ('20))

Fix any L and q large enough. For any high-energy change-point τ_k^*, we have

$$\mathbb{P}\left(d_{H,1}(\widehat{\boldsymbol{\tau}}, \tau_k^*)\mathbb{1}_{\mathcal{A}_q} \geq c\frac{x}{\Delta_k^2}\right) \lesssim e^{-x} \quad \forall x \geq 1 \ .$$

Remarks :

- Recovers the optimal subexponential rate of order Δ_k^{-2} for a specific change-point
- **Regional** to **Local** phenomenon :
  Detection= High-Energy **Localization** only depends on Δ_k !

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- Localization errors of high-energy change-points behave nearly **independently**.

Hausdorff and Wasserstein Loss

If $|\widehat{\boldsymbol{\tau}}| = |\boldsymbol{\tau}^*|$, define

$$d_W(\widehat{\boldsymbol{\tau}}, \boldsymbol{\tau}^*) = \sum_{k=1}^{K} |\widehat{\tau}_k - \tau_k^*|$$

$$d_H(\widehat{\boldsymbol{\tau}}, \boldsymbol{\tau}^*) = \max_{k=1}^{K} |\widehat{\tau}_k - \tau_k^*|$$

Corollary

Assuming that all change-points have high-energy, we deduce

$$\mathbb{E}\left[d_W\left(\widehat{\boldsymbol{\tau}}, \boldsymbol{\tau}^*\right)\mathbb{1}_{\mathcal{A}_q}\right] \lesssim \sum_{k=1}^{K}\left(e^{-c''\Delta_k^2} \wedge \frac{1}{\Delta_k^2}\right) ,$$

$$\mathbb{E}\left[d_H\left(\widehat{\boldsymbol{\tau}}, \boldsymbol{\tau}^*\right)\mathbb{1}_{\mathcal{A}_q}\right] \lesssim \max_{k\in\{1,\dots,K\}}\left(K e^{-c''\Delta_k^2} \wedge \frac{\log K}{\Delta_k^2}\right) .$$

Remark : Hausdorff and Wasserstein rates are minimax optimal.

Summary

Wrap-up :

- Regional to Local phenomenon.
- Low-energy change-points are (almost) unharmful.
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When $K = 1$, $\log(n/r_k)$ conditions are replaced by $\log\log(n/r_1)$ conditions.

Possible Extensions/ Open Questions :

- Heavier **tail** distribution, **time dependencies** :
  $\rightsquigarrow$ uniform control of the CUSUM (e.g.[Cho and Kirch, 2019])
- **Exact constant** for detection ?
- Similar results for two-steps bottom-up approach.

A general change-point framework

Data : Random sequence $Y = (y_1, y_2, \dots, y_n)$ in some measured space $\mathcal{Y}^n$.
Notation : $\mathbb{P}_i \in \mathcal{P}$ marginal distribution of y_i.
Change-point Functional : $\Gamma : \mathcal{P} \to \mathcal{V}$.

Change-Points : changes in the sequence $(\Gamma(\mathbb{P}_1), \Gamma(\mathbb{P}_2), \dots, \Gamma(\mathbb{P}_n))$

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Examples :

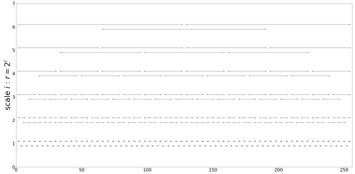
- **Gaussian mean Univariate change-point** :
  $\mathcal{Y} = \mathbb{R}$, $\mathcal{P} = \{\mathcal{N}(\theta, \sigma^2), \theta \in \mathbb{R}\}$, $\Gamma(\mathbb{P}) = \int x\mathbb{P}(dx)$
- **Gaussian mean multivariate change-point** [Chan and Chen, 2017, Wang and Samworth, 2018] :
  $\mathcal{Y} = \mathbb{R}^p$, $\mathcal{P} = \{\mathcal{N}(\theta, \sigma^2 \mathbf{I}_p), \theta \in \mathbb{R}^p\}$, $\Gamma(\mathbb{P}) = \int x\mathbb{P}(dx)$
- **Semi-parametric median (or quantile) univariate change-point** : [Jula Vanegas et al., 2021]
  $\mathcal{Y} = \mathbb{R}$, $\mathcal{P} = \{\text{Probability measure on } \mathbb{R}\}$, $\Gamma(\mathbb{P}) = median(\mathbb{P})$
- **Non-parametric univariate change-point** : [Padilla et al., 2019]
  $\mathcal{Y} = \mathbb{R}$, $\mathcal{P} = \{\text{Probability measure on } \mathbb{R}\}$, $\Gamma(\mathbb{P}) = \mathbb{P}$
- **Gaussian Covariance multivariate change-point** : [Wang et al., 2017]
  $\mathcal{Y} = \mathbb{R}^p$, $\mathcal{P} = \{\mathcal{N}(0, \Sigma), \Sigma \in \mathcal{S}_p^+\}$, $\Gamma(\mathbb{P}) = \int xx^T\mathbb{P}(dx)$

Goal : Detecting the change-points $\tau_1^*, \dots, \tau_K^*$
(leave aside the problem of localization)

A generic bottom-up algorithm

We are given :

- A collection (called a grid) $\mathcal{G}$ of (l, r) (location, scale) corresponding to segments $(l-r, l+r) \subset [1, n+1]$
  E.g. Complete Grid $\mathcal{G}_F = \{(l, r) : \ (l-r, l+r) \subset [1, n+1]\}$;
  Dyadic Grid $\mathcal{G}_D$.



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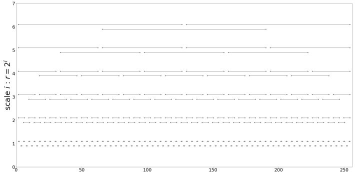

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Data: Local test $(T_{l,r})$
$\mathcal{CI} = \varnothing$; $\mathcal{CP} = \varnothing$;
For $r \in$ Scales
For $l \in$ Locations s.t. $T_{l,r} = 1$
if $[l-r+1, l+r-1] \cap \mathcal{CI} = \varnothing$
then
  $\mathcal{CI} \leftarrow \mathcal{CI} \cup [l-r+1, l+r-1]$;
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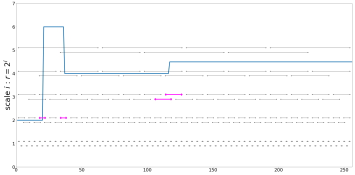
end
return $\widehat{\tau}_{ag} = \mathcal{CP}$



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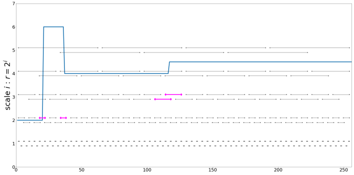

Dyadic grid

$O(n)$ tests for dyadic grids

Differs e.g. from [Chan and Chen, 2017, Kovács et al., 2020] because **intervals** of detected change-points are not allowed to intersect.

From multiple tests properties to detection properties

Proposition

1. *If* $FWER(\mathcal{T}) \leq \delta$*, then* $\widehat{\tau}_{ag}$ *satisfies* **(NoSp)** *with probability higher than* $1 - \delta$*.*
2. *Any change-point* τ_k^* *detected by a local test* $T_{l,\overline{r}_k}$ *with* $\overline{r}_k < r_k/4$*, is detected by* $\widehat{\tau}_{ag}$ *and*

$$d_{H,1}(\widehat{\tau}_{ag}, \tau_k^*) \leq \overline{r}_k - 1 \ .$$

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$\mathcal{K}^*$ = collection of **significant** change-points
$\asymp$ "with proba higher than $1 - \delta$, all $\tau_k^* \in \mathcal{K}^*$ detected by a suitable local test"

$\leadsto$ With probability higher than $1 - 2\delta$, $\widehat{\tau}_{ag}$ satisfies **NoSp** and **Detects** significant change-points.

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General recipe :

- Introducing a sensible notion of Energy
- Optimal testing with respect to that energy.
- Proper multiple testing correction to account for all tests in $\mathcal{G}_D$.

Gaussian Multivariate Change-point Model

$$y_i = \boldsymbol{\theta}_i + \epsilon_i, \qquad \text{where } \boldsymbol{\theta}_i \in \mathbb{R}^p \text{ and } \epsilon_i \overset{iid}{\sim} \mathcal{N}(0, \sigma^2 \mathbf{I}_p).$$

Objective : Detecting **times** $\tau_1^*, \ldots, \tau_K^*$ such that $\boldsymbol{\theta}_{\tau_k^*} \neq \boldsymbol{\theta}_{\tau_k^*-1}$
(side information $\boldsymbol{\theta}_{\tau_k^*} - \boldsymbol{\theta}_{\tau_k^*-1}$ is possibly sparse)

[Wang and Samworth, 2018, Chan and Chen, 2017, Enikeeva and Harchaoui, 2019, Liu et al., 2019]

Energy of a Change-Point

$$E_k^2 = r_k \frac{\|\boldsymbol{\theta}_{\tau_k^*} - \boldsymbol{\theta}_{\tau_k^*-1}\|^2}{\sigma^2}$$

Local Homogeneity tests on $[l-r, l+r)$
1st Simplification : two-sample tests over data in $[l-r, l)$ versus $[l, l+r)$.

2nd Simplification : (possibly-sparse) signal detection test with multivariate CUSUM statistics

$$\mathbf{C}_{l,r} = \left[\overline{\mathbf{Y}}_{[l,l+r)} - \overline{\mathbf{Y}}_{[l-r,l)}\right] \frac{\sqrt{2r}}{\sigma} \sim \mathcal{N}\left[\left(\overline{\boldsymbol{\theta}}_{[l,l+r)} - \overline{\boldsymbol{\theta}}_{[l-r,l)}\right) \frac{\sqrt{2r}}{\sigma}, \mathbf{I}_p\right]$$

Energy and Optimal Tests for sparse high-dimensional data

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Old toy detection Problem : [Baraud, 2002, Donoho and Jin, 2004, Collier et al., 2015]
$\rightsquigarrow$ Higher-Criticism + χ^2 type statistics (minimax optimal wrt sparsity s and p)

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$\rightsquigarrow$ Higher-Criticism + χ^2 type statistics (minimax optimal wrt sparsity s and p)

Not Sufficient : $\Omega(n)$ tests are considered
$\rightsquigarrow$ one also needs optimal dependencies wrt Types I and II error probabilities :
e.g. variants of HC [Liu et al., 2019] ; see also Pilliat et al.('20).

Optimal Detection

$\delta \in (0,1)$; s_k sparsity of change-point τ_k^*.

High-energy change-point

τ_k^* is a high-energy change-point if $E_k^2 \geq c\psi_{n,p,s_k,\delta}$ where

$$\psi_{n,p,s_k,\delta} = s_k \log\left(1 + \frac{\sqrt{p}}{s_k}\sqrt{\log\left(\frac{n}{r_k\delta}\right)}\right) + \log\left(\frac{n}{r_k\delta}\right) .$$

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With probability higher than $1-\delta$, $\widehat{\boldsymbol{\tau}}_{ag}$ *achieves* **(NoSp)** *and* **(Detects)** *all* **high-energy** *change-points* τ_k^* *with* $E_k^2 \geq c_+\psi_{n,p,s_k,\delta}$.

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Theorem (Pilliat et al.('20))

With probability higher than $1-\delta$, $\widehat{\boldsymbol{\tau}}_{ag}$ achieves **(NoSp)** *and* **(Detects)** *all* **high-energy** *change-points τ_k^* with $E_k^2 \geq c_+\psi_{n,p,s_k,\delta}$.*

Conversely, no procedure achieving **(NoSp)** *is able to* **(Detect)** *high-energy change-points τ_k^* with $E_k^2 \geq c_-\psi_{n,p,s_k,\delta}$*

Remark : Generalizes asymptotic result of [Chan and Chen, 2017]. For $K \leq 1$, see [Liu et al., 2019].

Application 2 : Covariance change-point detection

Gaussian covariance Change-point Model [Wang et al., 2017]

$$y_i \sim \mathcal{N}(0, \Sigma_i) \ , \quad i = 1, \ldots, n$$

Objective : Detecting **times** $\tau_1^*, \ldots, \tau_K^*$ such that $\Sigma_{\tau_k^*} \neq \Sigma_{\tau_k^*-1}$
Side information : $\max_i \|\Sigma_i\|_{op} \leq B^2$ (for some known B)

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Gaussian covariance Change-point Model [Wang et al., 2017]

$$y_i \sim \mathcal{N}(0, \Sigma_i) \ , \quad i = 1, \ldots, n$$

Objective : Detecting **times** $\tau_1^*, \ldots, \tau_K^*$ such that $\Sigma_{\tau_k^*} \neq \Sigma_{\tau_k^*-1}$

Side information : $\max_i \|\Sigma_i\|_{op} \leq B^2$ (for some known B)

Energy $E_k^2 := r_k \|\Sigma_{\tau_k^*} - \Sigma_{\tau_{k-1}^*}\|_{op}^2$.

Local test $T_{l,r} = \mathbb{1}\left\{\|\widehat{\Sigma}_{l,r} - \widehat{\Sigma}_{l,-r}\|_{op} \geq c_0 B^2 \left[\sqrt{\frac{p}{r}} + \frac{p}{r} + \sqrt{\frac{\log(\frac{2n}{\delta r})}{r}} + \frac{\log(\frac{2n}{\delta r})}{r}\right]\right\}$,

Optimal Detection

$\delta \in (0,1)$.

High-energy change-point

τ_k^* is a high-energy change-point if $E_k^2 \geq c\psi_{n,p,r_k}$ where

$$\psi_{n,p,r_k} := B^4 \left[\left(p + \log\left(\frac{2n}{r_k \delta} \right) \right) \wedge r_k \right]$$

Theorem (Pilliat et al.('20))

With probability higher than $1-\delta$, $\widehat{\boldsymbol{\tau}}_{ag}$ *achieves* **(NoSp)** *and* **(Detects)** *all* **high-energy** *change-points* τ_k^*, *with* $E_k^2 \geq c_+ \psi_{n,p,r_k}$

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Theorem (Pilliat et al.('20))

With probability higher than $1-\delta$, $\widehat{\boldsymbol{\tau}}_{ag}$ *achieves* **(NoSp)** *and* **(Detects)** *all* **high-energy** *change-points* τ_k^*, *with* $E_k^2 \geq c_+\psi_{n,p,r_k}$

Conversely, no procedure achieving **(NoSp)** *is able to* **(Detect)** *high-energy change-points* τ_k^* *with* $E_k^2 \geq c_-\psi_{n,p,r_k}$

(Improves over the $p\log(n)$ condition of [Wang et al., 2017])

Discussion

Main message

A simple reduction of change-point detection problems to multiple testing problems

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Sloppy optimality statement

If each test $T_{l,r}$ is **minimax optimal** (in some sense), then $\widehat{\boldsymbol{\tau}}_{ag}$ should (almost) optimally **detect** (in some sense).

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Open Questions

Localization rates require model-specific techniques.

For (sparse) high-dimensional change-points, there seem to exist several phase transitions from regional to local

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