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Normal Deviation of Gamma Processes in Random Media

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1. Introduction

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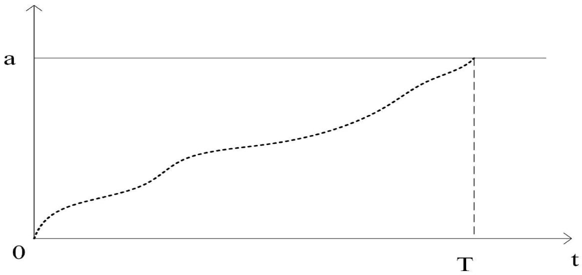
Introduction

A **gamma process** is a particular case of a **Lévy process**, see, e.g., Bertoin (1996), Sato (1999), and in fact it is a Markov process.

It is an independent and positive increments jump process. So, it is a good candidate for **monotone degradation phenomena**, and in fact it is extensively used in many applications, as in reliability/survival analysis, Singpurwalla (1995), Ting Lee and Whitmore (2023); in damage accumulation models, in cracking propagation in fracture mechanics, Lawless-Crowder (2004); in risk theory, Dufresne (1991), etc., where degradation of systems are observed.

A gamma process path, with threshold a and lifetime T



What is the purpose of this presentation?

The aim of this presentation is to approximate, by a diffusion process, the **deviation of a gamma processes from its average evolution** in a random environment.

In fact as the gamma process is an increasing one, the diffusion approximation requires an average approximation first.

This averaged process will serve as an equilibrium to the initial gamma process.

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Gamma process (I)

Gamma process as a Lévy process. Let us consider a Poisson random measure $\mu(du, ds)$ on $\mathbb{R}_+ \times \mathbb{R}_+$ with mean measure Π defined by

$$\Pi(du, ds) = bu^{-c-1}e^{-\alpha u}duds$$

where $\alpha > 0$, $b > 0$ and $c < 1$, are parameters. Define now the stochastic process $Z(t), t \geq 0$, by

$$Z(t) := \int_{\mathbb{R}_+ \times (0,t]} u\mu(du, ds), \quad t > 0, \tag{1}$$

which is an increasing process with Laplace transform

$$\mathbf{E}[e^{-\lambda Z(t)}] = \exp[-t \int_{(0,\infty)} (1 - e^{-\lambda u})\nu(du)] \tag{2}$$

where $\nu(du)ds = \Pi(du, ds)$.

Gamma process

We suppose now that $c = 0$, then the process $Z(t), t \geq 0$, defined by (1), is said to be a gamma process. This process is a particular case of a Lévy process, with Lévy measure,

$$\nu(du) = bu^{-1}e^{-\alpha u}du, \qquad u > 0,$$

which satisfy the following integrability condition

$$\int_{(0,\infty)} (1 \wedge x)\nu(dx) < \infty. \tag{3}$$

It is worth noticing that this integral, in our case here, is bounded by $1/\alpha$. So, the measure ν is σ-finite. Such a gamma process, $Z(t)$, increases by jumps only, and, as $\nu(0,\infty) = +\infty$, the set of mass points $\{s_i\}$ is dense in $\mathbb{R}_+$, and has infinitely many jumps in every open interval (s,t), $s < t$.

Gamma process (II)

A process $(Z(t), t \geq 0)$ is said to be an homogeneous gamma process, denoted by $GP(bt, \alpha)$, if the following conditions are fulfilled:

1. $Z(0) = 0$;
2. Z has independent increments;
3. The increments $Z(t+h) - Z(t)$, for $t \geq 0, h > 0$, are stationary and follow the gamma distribution, i.e., its pdf f_h is the $Ga(bh, \alpha)$, see (4) below.

The Law of Gamma process

The name of gamma process comes from the fact that the law of $Z(t)$ is the gamma distribution, $Ga(bt, \alpha)$, i.e.,

$$f_t(x) = \frac{(\alpha x)^{bt-1}}{\Gamma(bt)} \alpha e^{-\alpha x}, \quad x \geq 0, \tag{4}$$

with shape parameter bt and scale or rate parameter $\alpha > 0$, where

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

is the gamma Eulerian function defined for $x > 0$.
Consequently, with mean bt/α and variance bt/α^2,

Gamma process (III)

A gamma process is also defined by the following infinitesimal generator, L,

$$L\varphi(u) = \int_E [\varphi(u+v) - \varphi(u)]\nu(dv)$$

of its semigroup operators, $(T_t, t \geq 0)$,

$$T_t\varphi(u) = \mathbb{E}[\varphi(u + Z(t))]$$

for $\varphi \in C_0(\mathbb{R}_+)$, the continuous functions defined on $\mathbb{R}_+$ which vanish at infinity.

The LT of Gamma process

The Laplace transform of $Z(t)$ is

$$\mathbb{E}[e^{-\lambda Z(t)}] = (\frac{\alpha}{\lambda+\alpha})^{bt} = \exp\left[-t\int_{(0,\infty)}(1-e^{-\lambda x})\nu(dx)\right]$$

with $\lambda \geq 0$, and

$$\nu(dx) = \frac{be^{-\alpha x}}{x}dx$$

for $x > 0$, is the Lévy measure.

Markov process as Environment

Let us consider a regular pure jump Markov process X, with state space the measurable space $(E, \mathcal{E})$, with countably generated σ-algebra. We suppose also that X is cadlag, i.e., continuous on the right having left limits at any point of time $t > 0$.

Denote by $0 = S_0 < S_1 < \ldots$ the jump times and $J_n, n \geq 0$, the successive visited states.
Define also the r.v.s $W_{n+1} := S_{n+1} - S_n$, for $n \geq 0$, $W_0 = 0$.

As usually, we denote by $\mathbf{P}_x(\cdot)$ the conditional probability $\mathbf{P}(\cdot \mid J_0 = x)$ and by $\mathbf{E}_x$ the corresponding expectation operator.

The Markov process, $X(t), t \geq 0$, is defined by its generator, A,

$$A\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)], \qquad (5)$$

where $q(x), x \in E$, is the intensity function of jumps, and P is transition kernel of the embedded Markov chain (J_n) of the Markov process X.

We consider that the Markov process X is uniformly ergodic, with ergodic probability π.

Let Π be the stationary projector operator of the above Markov process, X, i.e.,

$$\Pi\varphi(x) = \int_E \pi(dv)\varphi(v)\mathbf{1}_E(x).$$

Define the transition probability

$$P_t(x, B) := \mathbb{P}(X(t) \in B \mid X(0) = x)$$

and the transition operator P_t, $t \geq 0$.

The potential operator R_0 of P_t is defined by

$$R_0 = \int_0^\infty (P_t - \Pi) dt$$

and

$$AR_0 = R_0 A = \Pi - I.$$

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Gamma process in Random environment

Let us consider now a right continuous Markov process, say $X(t), t \geq 0$, with state space $(E, \mathcal{E})$, which represent the random media or equivalently the random environment.

Consider also a gamma process, say $Z(t), t \geq 0$, i.e., $GP(bt, \alpha)$, defined by its Laplace transform

$$\mathbb{E}[e^{-\lambda Z(t)}] = \exp\Big[-t \int_{(0,\infty)} (1 - e^{-\lambda u}) b u^{-1} e^{-\alpha u} du \Big]$$

where $\nu(du) = bu^{-1}e^{-\alpha u}du$ is the Lévy measure which satisfy the above condition (3).

For the gamma process, the shape and rate parameters, at time $t \geq 0$, are now functions of the states $x \in E$, i.e.,

$$b := b(x), \quad \text{and} \quad \alpha := \alpha(x)$$

respectively, on the event $\{X(t) = x\}$.

Consequently, the Lévy measure, ν, depends also on the state $x \in E$, i.e.,

$$\nu(x, du) = b(x) u^{-1} e^{-\alpha(x)u} du.$$

Define also the natural filtration $\mathcal{F}_t := \sigma(X(s), 0 \leq s \leq t)$, for $t \geq 0$.

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Assumptions

The following assumption are important in the sequel.

A1: We suppose here that for any fixed $x \in E$, the gamma process $Z(t)$, $GP(b(x)t, \alpha(x))$, and the process $X(t)$ are independent.
We suppose moreover that $\kappa := \inf\{\alpha(x) : x \in E\} > 0$.

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A2: We assume that

$$\int_E \pi(dx)b(x)/\alpha(x) < \infty.$$

The Gamma process in Random Media LT

Proposition

Under assumption A1, we have

$$\mathbf{E}_x[e^{-\lambda Z(t)} \mid \mathcal{F}_t] = \exp\left[-\int_0^t b(X(s))\ln\left(1+\frac{\lambda}{\alpha(X(s))}\right)ds\right] \quad (6)$$

for $0 \leq \lambda \leq \kappa$.

From Equation (6), it is not difficult to show that

$$\mathbf{E}_x[e^{-\lambda Z(t)} \mid \mathcal{F}_t] = \exp\left[-\int_0^t b(X(s))ds \int_{(0,\infty)} \frac{1-e^{-\lambda u}}{u} e^{-\alpha(X(s))u} du\right]. \quad (7)$$

Average approximation

Let us consider now the above formulation into a series scheme, for $\varepsilon > 0$, the series parameter as follows.

Define the Markov process in rescaled time $X^\varepsilon(t) = X(t/\varepsilon)$, and the natural filtration $\mathcal{F}_t^\varepsilon := \sigma(X^\varepsilon(s), 0 \leq s \leq t)$, for $t \geq 0$, $\varepsilon > 0$.

Now define the family of processes

$$Y^\varepsilon(t) := Y^\varepsilon(t, \lambda, X) := \mathbf{E}[e^{-\lambda Z^\varepsilon(t)} \mid \mathcal{F}_t^\varepsilon] \quad (8)$$

and the function $a(x;\lambda) := b(x)\ln(1 + \frac{\lambda}{\alpha(x)})$, $x \in E$, $\lambda > 0$.

Average approximation

THEOREM

Under conditions A1-A2, the following weak convergence holds

$$Y^{\varepsilon}(t) \Rightarrow \exp(-\hat{a}(\lambda)t), \quad \textit{as} \quad \varepsilon \downarrow 0,$$

where $\hat{a}(\lambda) := \int_E \pi(dx)\alpha(x;\lambda)$*, and the limit is the Laplace transform of an independent increment process.*

A key result

Lemma

The function $\exp(-\hat{a}(\lambda)t)$*, for any fixed* $t > 0$*, is the Laplace transform of a non negative infinitely divisible r.v.*

Consequently, the process is of independent increments.

Let us now consider the processes

$$S^{\varepsilon}(t) := \int_0^t a(X(s/\varepsilon^2); \lambda) ds.$$

Normal deviation

Let us define the family of stochastic processes

$$V^\varepsilon(t) := \exp\Big[-S^\varepsilon(t)\Big], \quad t \geq 0, \quad \varepsilon > 0. \tag{9}$$

In the usual diffusion approximation scheme we have to consider an additional condition, the balance condition, which has to be:

$$\hat{a}(\lambda) := \Pi a(\cdot;\lambda) = 0.$$

But now we have $\hat{a}(\lambda) > 0$.

In that case, we can obtain a diffusion approximation by equilibrate the above process by the averaged one. So, we have a diffusion approximation with equilibrium or a normal deviation theorem as follows.

THEOREM

Under conditions A1-A2, the processes

$$\varepsilon^{-1}(S^{\varepsilon}(t) - \hat{a}(\lambda)t), t \geq 0, \varepsilon > 0,$$

converges weakly, as $\varepsilon \downarrow 0$, *to the process* $\sigma(\lambda)W(t)$, *where* $W(t), t \geq 0$, *is a standard Wiener process, and*

$$\sigma^2(\lambda) := 2\int_E \pi(dx)a(x,\lambda)R_0 a(x,\lambda).$$

Moreover, this result can also be written as follows

$$\varepsilon^{-1}(-\ln V^{\varepsilon}(t) - \hat{a}(\lambda)t) \Rightarrow \sigma(\lambda)W(t), \qquad \varepsilon \downarrow 0.$$

Elements of Proofs

The proof of the previous theorem is as follows.

$$\varepsilon^{-1}(S^{\varepsilon}(t) - \hat{a}(\lambda)t) = \varepsilon^{-1}\int_0^t [a(X(s/\varepsilon^2);\lambda) - \hat{a}(\lambda)]ds$$

Consider the coupled processes $S^{\varepsilon}(t)$, $X(t/\varepsilon^2)$, $t \geq 0$, $\varepsilon > 0$. Their generators are

$$L^{\varepsilon}\varphi(u,x) = [\varepsilon^{-2}A + \varepsilon^{-1}B]\varphi(u,x)$$

where A is the generator of the Markov process X and B is the generator of the random evolution defined by

$$B(x;\lambda)\varphi(u) = a(x;\lambda)\varphi'(u).$$

Now, by the solution of the singular perturbation problem, on the test functions $\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x) + \varepsilon^2\varphi_2(u,x)$, i.e.,

$$L^\varepsilon\varphi^\varepsilon(u,x) = L\varphi(u) + \varepsilon\theta^\varepsilon(u,x),$$

where $\theta^\varepsilon(u,x)$ is a negligible, we obtain the limit operator L,

$$L\varphi(u) = \Pi B(x;\lambda) R_0 B(x;\lambda)\Pi\varphi(u)$$

from which we obtain directly

$$L\varphi(u) = \frac{1}{2}\sigma^2(\lambda)\varphi''(u).$$

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Concluding remarks

- The averaging and normal deviation results give us the possibility to obtain estimation and confidence intervals for gamma processes in random media and its applications.
- In order to use the above results in practical application some additional work remains to perform concerning inversion of Laplace transforms.
- Different stochastic environments can be considered, as semi-Markov, diffusion, etc.

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