Optimal guaranteed estimation methods for the Cox-Ingersoll-Ross (CIR) models

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General model and sampling Method

$$\mathrm{d}X_t = b(\theta_0, X_t)\mathrm{d}t + \sigma(\theta_1, X_t)\mathrm{d}W_t.$$

- **Problem**: Using data observed from a single trajectory of X estimate parameters $\theta_0 \in \Theta_0 \subset \mathbb{R}^{d_0}$ and $\theta_1 \in \Theta_1 \subset \mathbb{R}^{d_1}$
- Continuous sampling: $(X_t)_{t\in[0,T]}$
- Discrete sampling: $(X_{k\Delta_n})_{0\leq k\leq n}$
  - High frequency data in a fixed period: $\Delta_n \to 0$ and $n\Delta_n = T$ fixed
  - Low frequency data in a long period: $\Delta_n = \Delta$ fixed, and $n\Delta_n \to \infty$
  - High frequency data in a long period: $\Delta_n \to 0$ and $n\Delta_n \to \infty$
- The estimation of θ_1 which is called volatility in mathematical finance, is based on the fact that the quadratic variation $\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}})^2 \to \int_0^T \sigma(\theta_1, X_s)^2 \mathrm{d}s.$
- For any fixed T, θ_1 can be consistently estimated if $\Delta_n \to 0$.
- In the following, we suppose that θ_1 is known and try to estimate θ_0

Maximum likelihood estimation

In Liptser & Shiryaev'01 and Kutoyants'04, Radon-Nikodym derivative of the probability measure induced by

$$\mathrm{d}X_t = b(\theta, X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}$$

with $\theta \in \Theta \subset \mathbb{R}^d$ is given and the likelihood function is

$$L_T(\theta) = \exp\left(\int_0^T \frac{b(\theta, X_t) - b(\theta_0, X_t)}{\sigma^2(X_t)}\mathrm{d}X_t - \frac{1}{2}\int_0^T \frac{b^2(\theta, X_t) - b^2(\theta_0, X_t)}{\sigma^2(X_t)}\mathrm{d}t\right)$$

for any fixed $\theta_0 \in \Theta \subset \mathbb{R}^d$. The MLE $\widehat{\theta}_T$ of θ is defined by

$$\widehat{\theta}_T = \mathrm{argmax}_{\theta \in \Theta} L_T(\theta)$$

The consistency and asymptotic normality of $\widehat{\theta}_T$ have been shown under some regularity conditions of b and σ, generally in ergodic case.

Ergodicity

Kutoyants'04

- When the scale function $V(x) = \int_0^x \exp\left(-2\int_0^y \frac{b(\theta,v)}{\sigma^2(v)}\mathrm{d}v\right)\mathrm{d}y$ tends to $+\infty$ and $-\infty$ when x tends to $+\infty$ and $-\infty$, the process (X_t) is recurrent.
- Moreover, (X_t) is positively recurrent when $G = \int_{-\infty}^{+\infty} \frac{1}{\sigma^2(y)} \exp\left(2\int_0^y \frac{b(\theta,v)}{\sigma^2(v)}\mathrm{d}v\right)\mathrm{d}y < \infty$.
- In this case (X_t) is an ergodic process, i.e., the Law of large numbers holds that

$$\frac{1}{T}\int_0^T h(X_s)ds \longrightarrow \int h(x)f(x)\mathrm{d}x \quad a.s.$$

for any measurable integrable function h where

$$f(x) = \frac{1}{G\sigma^2(x)} \exp\left(2\int_0^x \frac{b(\theta,v)}{\sigma^2(v)}\mathrm{d}v\right)$$

is the stationary density.

The sequential analysis approaches

- The first approach: in Liptser & Shiryaev'01 and Novikov'72 for **the scalar Ornstein-Uhlenbeck process in continuous time**

$$\mathrm{d}X_t = \theta X_t \mathrm{d}t + \mathrm{d}W_t. \tag{1}$$

The Maximum Likelihood Estimator (MLE) for the parameter θ defined as

$$\widehat{\theta}_T = (\int_0^T X_s^2 \mathrm{d}s)^{-1} \int_0^T X_s \mathrm{d}X_s \,.$$

Then the Sequential Maximum Likelihood Estimator (SMLE) for the parameter θ defined as

$$\widehat{\theta}_{\tau_H^*} = \frac{1}{H} \int_0^{\tau_H^*} X_s \mathrm{d}X_s \,,$$

with the stopping time

$$\tau_H^* = \inf \left\{ t \geq 0 \, : \, \int_0^t X_s^2 \mathrm{d}s \geq H \right\} \,,$$

where $H > 0$ is some fixed non random arbitrary constant. One can check that $\widehat{\theta}_{\tau_H^*}$ is $\mathcal{N}(\theta, H^{-1})$ for any $H > 0$.

The sequential analysis approaches

- Discrete time: Borisov & Konev'77 (non-asymptotic), Lai & Siegmund'83 (asymptotic).
- *Guaranteed* two-step estimation method for multidimensional parameter case: Konev & Pergamenshchikov'81(1)(2), Konev & Pergamenshchikov'88, Galtchouk & Konev'01.
- Guaranteed estimation property in the non-asymptotic setting with dependent observations: Konev & Pergamenshchikov'97.
- For the proposed sequential estimation methods, the asymptotic properties, as $H \to \infty$, were studied (see, e.g., in Konev & Pergamenshchikov'84'85'86, Pergamenshchikov'85).
- Truncated versions of developed sequential estimators were proposed in Konev & Pergamenshchikov'90'92'97.

Parameter estimation problems for the CIR processes

We consider the stochastic differential equation

$$\mathrm{d}X_t = (a - bX_t)\mathrm{d}t + \sqrt{\sigma X_t}\mathrm{d}W_t, \quad X_0 = x > 0\,, \tag{2}$$

where $a > 0$, $b \in \mathbb{R}$ and $\sigma > 0$ and $(W_t)_{t\geq 0}$ is a standard Brownian motion.

- For $\theta = b$, the MLE (see, e.g., in Ben Alaya & Kebaier'12) is defined as $\widehat{\theta}_T = \frac{aT - X_T + x}{\int_0^T X_s \mathrm{d}s}$.
- For $\theta = a$, the MLE is given as $\widehat{\theta}_T = \frac{bT + \int_0^T X_t^{-1}\mathrm{d}X_t}{\int_0^T X_t^{-1}\mathrm{d}t}$.
- For $\theta = (a, b)$, the MLE (see, e.g., in Ben Alaya & Kebaier'13) is defined as

$$\widehat{\theta}_T = \begin{cases} \widehat{a}_T & = & \frac{\int_0^T X_t \mathrm{d}t \int_0^T X_t^{-1}\mathrm{d}X_t - T(X_T - x)}{\int_0^T X_t \mathrm{d}t \int_0^T X_t^{-1}\mathrm{d}t - T^2} \\ \widehat{b}_T & = & \frac{T\int_0^T X_t^{-1}\mathrm{d}X_t - (X_T - x)\int_0^T X_t^{-1}\mathrm{d}t}{\int_0^T X_t \mathrm{d}t \int_0^T X_t^{-1}\mathrm{d}t - T^2} \end{cases}$$

- These papers show the asymptotique behavior of the error $(\widehat{\theta}_T - \theta)$ for ergodic and non-ergodic cases.
- There are still no results on a guaranteed estimation for the CIR process.

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Case $\theta = a$ I

We define the sequential estimation procedure $\delta_H^* = (\tau_H^*, \theta_H^*)$ with $H > 0$ for the parameter a as

$$\tau_H^* = \inf\left\{t : \int_0^t X_s^{-1} \mathrm{d}s \geq H\right\} \quad \text{and} \quad \theta_H^* = \frac{b\tau_H^* + \int_0^{\tau_H^*} X_s^{-1} \mathrm{d}X_s}{H}.$$

First we study non asymptotic properties of this procedure, i.e. for any fixed threshold $H > 0$.

Theorem 1 (Ben Alaya, N. and Pergamenchtchikov'23)

For any $\boxed{b \geq 0, \; a > 0}$ and for any fixed $H > 0$ the sequential procedure δ_H^* possesses the following properties:

1. $\mathbf{P}_\theta(\tau_H^* < \infty) = 1$;
2. the sequential estimator θ_H^* is normally distributed with parameters

$$\mathbf{E}_\theta \theta_H^* = a \quad \text{and} \quad \mathbf{E}_\theta(\theta_H^* - a)^2 = \frac{\sigma}{H}.$$

+ When $a < \sigma/2$, $\int_0^t X_s^{-1/2} \mathrm{d}W_s$ is not defined for any fixed non random $t > 0$. Therefore, the non sequential MLE can not be calculated for this case, but the sequential procedure is well defined.

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Case $\theta = b$

We define the sequential procedure $\delta_H^* = (\tau_H^*, \theta_H^*)$ with $H > 0$ for the parameter b as

$$\tau_H^* = \inf \left\{ t : \int_0^t X_s \mathrm{d}s \geq H \right\} \quad \text{and} \quad \theta_H^* = \frac{a\tau_H^* - X_{\tau_H^*} + x}{H} \,,$$

First we study non asymptotic properties of this procedure, i.e. for any fixed threshold $H > 0$.

Theorem 2 (Ben Alaya, N. and Pergamenchtchikov'23)

For any $\boxed{a > 0, \;\; b \in \mathbb{R}}$ and for any fixed $H > 0$, δ_H^* possesses the following properties:

1. $\mathbf{P}_\theta(\tau_H^* < \infty) = 1\,;$
2. the sequential estimator θ_H^* is normally distributed with parameters

$$\mathbf{E}_\theta \theta_H^* = b \quad \text{and} \quad \mathbf{E}_\theta(\theta_H^* - b)^2 = \frac{\sigma}{H} \,.$$

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Case $\theta = (a, b)$

We rewrite the CIR model as

$$\mathrm{d}X_t = \mathbf{g}_t^\top \theta \mathrm{d}t + \sqrt{\sigma X_t}\mathrm{d}W_t \,,$$

where $\mathbf{g}_t = (1\,,\,-X_t)^\top$. In this section we assume, that $b > 0$ and $a > \sigma/2$. Then, in view of the results from Ben Alaya & Kebaier'13, the random matrix

$$G_t = \int_0^t X_s^{-1}\mathbf{g}_s\mathbf{g}_s^\top \mathrm{d}s = \begin{pmatrix} \int_0^t X_s^{-1}\mathrm{d}s & -t \\ -t & \int_0^t X_s ds \end{pmatrix}$$

possesses the following asymptotic property

$$\lim_{t\to\infty} \frac{1}{t} G_t = F = \begin{pmatrix} f_1 & -1 \\ -1 & f_2 \end{pmatrix} \qquad \mathbf{P}_\theta - \quad \text{a.s.}\,,$$

where $f_1 = 2b/(2a - \sigma)$ and $f_2 = a/b$. Here, F is positively definite matrix.

We use the two-step sequential fixed accuracy estimation method developed in Konev & Pergamenshchikov'81'85

- First step: we construct the sequence of the sequential procedures $\left(\delta_n = (\mathbf{t}_n, \widehat{\theta}_{\mathbf{t}_n})\right)_{n\geq 1}$. We fix a non random sequence of non-decreasing positive numbers $(\kappa_n)_{n\geq 1}$ for which

$$\rho = \sum_{n\geq 1} \frac{1}{\kappa_n} < \infty \,. \tag{3}$$

Now for any $z > 0$ we set

$$\mathbf{t}_z = \inf\left\{t \geq 0 : \int_0^t X_s^{-1} \, |\mathbf{g}_s|^2 \mathrm{d}s \geq z\right\} , \tag{4}$$

Let $\mathbf{t}_n = \mathbf{t}_{\kappa_n}$ and define the sequential MLE as

$$\widehat{\theta}_{\mathbf{t}_n} = G_{\mathbf{t}_n}^{-1} \int_0^{\mathbf{t}_n} X_s^{-1} \mathbf{g}_s \mathrm{d}X_s \tag{5}$$

- $\boxed{\text{Second step}}$: we construct a sequential aggregation estimation procedure which is defined as weighted sum of the estimators (5). First we set

$$\mathbf{b}_n = \frac{1}{|G_{\mathbf{t}_n}^{-1}|\,\kappa_n} \mathbf{1}_{\{\lambda_{min}(G_{\mathbf{t}_n})>0\}}$$

where $|G|^2 = \operatorname{tr} GG^\top$, and we define the stopping time as

$$\upsilon_H^* = \inf\left\{k \geq 1 : \sum_{n=1}^{k} \mathbf{b}_n^2 \geq H\right\}, \qquad (6)$$

for a positive non random threshold $H > 0$. We define the sequential estimator as

$$\theta_H^* = \left(\sum_{n=1}^{\upsilon_H^*} \mathbf{b}_n^2\right)^{-1} \sum_{n=1}^{\upsilon_H^*} \mathbf{b}_n^2\, \widehat{\theta}_{\mathbf{t}_n}. \qquad (7)$$

So, we obtain aggregated two-step sequential procedure

$$\boxed{\delta^*_H = \left(\tau^*_H\,,\,\theta^*_H\right) \quad \text{and} \quad \tau^*_H = \mathbf{t}_{\upsilon^*_H}}. \tag{8}$$

Theorem 3 (Ben Alaya, N. and Pergamenchtchikov'23)

For any $b > 0$ and $a > \sigma/2$ and for any $H > 0$ the procedure (8) has the following properties

$$\tau^*_H < +\infty \quad \mathbf{P}_\theta - \text{a.s.}$$

and

$$\mathbf{E}_\theta \left|\theta^*_H - \theta\right|^2 \le \rho \frac{\sigma}{H}\,,$$

where the coefficient ρ is defined in (3).

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LAN property

A family of probability measures $(\mathbf{P}_{\theta,T})_{\theta\in\Theta,T>0}$ with $\Theta \subseteq \mathbb{R}^k$ is called to satisfy the Local Asymptotic Normality condition (LAN) at a point $\theta_0 \in \Theta$ if there exist scale $k \times k$ matrix $\varphi_T(\theta_0)$ going to zero as $T \to \infty$ such that for any $u \in \mathbb{R}^k$ for which the point $\widetilde{\theta} = \theta_0 + \varphi_T(\theta_0)u$ belongs to Θ, the Radon-Nikodym derivative has the following asymptotic representation

$$\ln \frac{\mathrm{d}\mathbf{P}_{\widetilde{\theta},T}}{\mathrm{d}\mathbf{P}_{\theta_0,T}} = u^\top \xi_T - \frac{|u|^2}{2} + \mathbf{r}_T(u)\,,$$

$$\xi_T \xrightarrow[T\to\infty]{\mathcal{L}(\mathbf{P}_{\theta_0,T})} \mathcal{N}(0,J) \quad \text{and} \quad \mathbf{r}_T(u) \xrightarrow[T\to\infty]{\mathbf{P}_{\theta_0,T}} 0\,.$$

where J is the identity matrix.

From now, we take the parameter set

$$\Theta \subseteq \{(a,b)\,:\, a>\sigma/2\,,\; b>0\} =]\sigma/2\,,+\infty[\times]0,+\infty[.$$

Consequences of the LAN property

When the LAN property holds at $\theta_0 \in \Theta$ with $\varphi_T(\theta_0)$

- Convolution theorem: An estimator $\widehat{\theta}_T$ of θ is asymptotically efficient at θ_0 in the sense of Hájek-Le Cam : we have a central limit theorem for the error with rate $\varphi_T^{-1}(\theta_0)$.
- Minimax theorem: $I(\theta_0)^{-1}$ gives a lower bound for the quadratic risque of any estimator of θ_0. For the class of unbaised estimators it is the Cramér-Rao lower bound.
- References: *Le Cam'60 & Hájek'70,72* (LAN).

Minimax inequality

Let now $\theta_0 \in \Theta$ and $\gamma > 0$ such that $\{|\theta - \theta_0| \le \gamma\} \subseteq \Theta$. We denote by $\mathcal{H}_T(\theta_0, \gamma)$ the local class of sequential procedures $\delta_T = (\tau\,, \widehat{\theta}_\tau)$ such that

$$\boxed{\sup_{|\theta-\theta_0|<\gamma} \mathbf{E}_\theta \tau \le T}.$$

Inspired by the ideas from Corollary 2 in Efroimovich'80, we prove the following proposition.

Proposition (Ben Alaya, N. and Pergamenchtchikov'23)

Assume that, LAN holds for θ_0 from $\Theta \subset \mathbb{R}^k$ with the function $\varphi_T = (I(\theta_0)T)^{-1/2}$ and $I(\theta_0)$ is the Fisher information matrix. Then, for any $\gamma > 0$ for which $\{|\theta - \theta_0| \le \gamma\} \subseteq \Theta$,

$$\lim_{T\to\infty} \inf_{\delta\in\mathcal{H}_T(\theta_0,\gamma)} \sup_{|\theta-\theta_0|<\gamma} \mathbf{E}_\theta \left|\varphi_T{}^{-1}\,(\widehat{\theta}_\tau - \theta)\right|^2 \ge k\,.$$

Now we need to compare the defined sequential procedure $\delta_H^* = (\tau_H^*, \theta_H^*)$ with other sequential procedure. To this end we set

$$\Xi_H = \left\{ \delta = (\tau, \widehat{\theta}_\tau) \, : \, \sup_{\theta \in \Theta} \frac{\mathbf{E}_\theta \tau}{\mathbf{E}_\theta \tau_H^*} \le 1 \right\} .$$

Now we obtain a lower bound for this class.

Theorem 4 (Ben Alaya, N. and Pergamenchtchikov'23)

Let θ_0 from $\Theta \subset \mathbb{R}^k$ such that $\{|\theta - \theta_0| < \gamma\} \subset \Theta$ for all sufficiently small $\gamma > 0$. Assume that, LAN holds in θ_0 with the normalizing function $\varphi_T = (I(\theta_0)T)^{-1/2}$ and $I(\theta_0)$ is the Fisher information matrix. Then,

$$\lim_{H \to \infty} \inf_{\delta \in \Xi_H} \sup_{\theta \in \Theta} \mathbf{E}_\theta \left| \upsilon_H(\theta)^{1/2} (\widehat{\theta}_\tau - \theta) \right|^2 \ge k \, ,$$

where $\upsilon_H(\theta) = I(\theta) \mathbf{E}_\theta \tau_H^*$.

Optimality of SMLE θ^*_H for $\theta = a$, $k = 1$

Theorem 5 (Ben Alaya, N. and Pergamenchtchikov'23)

For any $\boxed{b > 0}$, any compact set $\boxed{\Theta \subset]\sigma/2, +\infty[}$ and for any $r > 0$

$$\lim_{H\to\infty} \sup_{\theta\in\Theta} \mathbf{E}_\theta \left| \frac{\tau^*_H}{H} - I_0^{-1}(\theta) \right|^r = 0 \,,$$

where $I_0(\theta) = 2b/(2a - \sigma)$.

In this case:

- $\upsilon_H(\theta) = \sigma^{-1}(2\theta - \sigma)^{-1}\, 2b\, \mathbf{E}_\theta \tau^*_H \approx \sigma^{-1} H$ as $H \to \infty$.
- $\lim_{H\to\infty} \sup_{\theta\in\Theta} \upsilon_H(\theta)\, \mathbf{E}_\theta \left(\theta^*_H - \theta\right)^2 = 1.$

Optimality of SMLE θ^*_H for $\theta = a$, $k = 1$

Theorem 6 (Ben Alaya, N. and Pergamenchtchikov'23)

For any $b > 0$ and compact set $\Theta \subset]\sigma/2, +\infty[$ the sequential procedure $\delta^*_H = (\tau^*_H, \theta^*_H)$ is asymptotically optimal in the minimax setting, i.e.

$$\lim_{H\to\infty} \sigma^{-1} H \inf_{(\tau,\widehat{\theta}_\tau)\in\Xi_H} \sup_{\theta\in\Theta} \mathbf{E}_\theta \left(\widehat{\theta}_\tau - \theta\right)^2 = \lim_{H\to\infty} \sigma^{-1} H \sup_{\theta\in\Theta} \mathbf{E}_\theta \left(\theta^*_H - \theta\right)^2 = 1.$$

Optimality of SMLE θ^*_H for $\theta = b$, $k = 1$

Theorem 7 (Ben Alaya, N. and Pergamenchtchikov'23)

For any $\boxed{a > \sigma/2}$, any compact set $\boxed{\Theta \subset]0, +\infty[}$ and any $r > 0$

$$\lim_{H\to\infty} \sup_{\theta\in\Theta} \mathbf{E}_\theta \left| \frac{\tau^*_H}{H} - \frac{1}{I_0(\theta)} \right|^r = 0\,,$$

where $I_0(\theta) = a/\theta$.

In this case:

- $\upsilon_H(\theta) = \sigma^{-1}\theta^{-1} a\, \mathbf{E}_\theta \tau^*_H \approx \sigma^{-1} H$ as $H \to \infty$.
- $\lim_{H\to\infty} \sup_{\theta\in\Theta} \upsilon_H(\theta)\, \mathbf{E}_\theta \left(\theta^*_H - \theta\right)^2 = 1$.

Optimality of SMLE θ^*_H for $\theta = b$, $k = 1$

Theorem 8 (Ben Alaya, N. and Pergamenchtchikov'23)

For any $a > \sigma/2$ and any compact set $\Theta \subset]0, +\infty[$ the sequential procedure $\delta^*_H = (\tau^*_H, \theta^*_H)$ is asymptotically optimal in the minimax setting, i.e.

$$\lim_{H\to\infty} \sigma^{-1} H \inf_{(\tau,\widehat{\theta}_\tau)\in\Xi_H} \sup_{\theta\in\Theta} \mathbf{E}_\theta \left(\widehat{\theta}_\tau - \theta\right)^2 = \lim_{H\to\infty} \sigma^{-1} H \sup_{\theta\in\Theta} \mathbf{E}_\theta \left(\theta^*_H - \theta\right)^2 = 1.$$

Optimality of SMLE θ^*_H for $\theta = (a, b)$, $k = 2$

We chose the sequence $(\kappa_n)_{n\geq 1}$ as follows

$$\kappa_n = \begin{cases} H\,, & \text{for} \quad n \leq \mathbf{n}^*_H\,; \\ \kappa^*_n\,, & \text{for} \quad n > \mathbf{n}^*_H\,, \end{cases}$$

where $\mathbf{n}^*_H = L_H H$ and $L_H \geq 1$ is slowly increasing function, i.e.

$$\lim_{H\to\infty} L_H = +\infty \quad \text{and} \quad \lim_{H\to\infty} \frac{L_H}{H^\delta} = 0 \quad \text{for any} \quad \delta > 0\,.$$

Moreover, $(\kappa^*_n)_{n\geq 1}$ is a sequence of positive increasing numbers such, that for some $\mu > 1$ and $0 < \varrho < 1$,

$$\limsup_{n\to\infty} n^{-\mu}\,\kappa^*_n < \infty \quad \text{and} \quad \limsup_{n\to\infty} n^{-\varrho} \sum_{k=1}^{n} \frac{1}{\sqrt{\kappa^*_k}} < \infty\,.$$

For example, we can take $\mathbf{n}^*_H = H \ln H$ and $\kappa^*_n = n^\mu$ for some $\mu > 1$.

Optimality of SMLE θ^*_H for $\theta = (a, b)$, $k = 2$

Theorem 9 (Ben Alaya, N. and Pergamenchtchikov'23)

For any compact set $\Theta \subset]\sigma/2, +\infty[\times]0, +\infty[$ for the duration time in the sequential procedure (8) we have for any $r > 0$

$$\lim_{H\to\infty} \sup_{\theta\in\Theta} \mathbf{E}_\theta \left| \frac{\tau^*_H}{H} - \frac{1}{\mathrm{tr} F} \right|^r = 0\,,$$

where the matrix F is defined above by $\lim_{t\to\infty} \frac{1}{t} G_t = F$.

- Let $\tilde{F} = F/tr(F)$. In this case $\upsilon_H(\theta) = \sigma^{-1} F\, \mathbf{E}_\theta \tau^*_H \approx \sigma^{-1} H \tilde{F}$ as $H \to \infty$.
- $\lim_{H\to\infty} \sup_{\theta\in\Theta}\ \mathbf{E}_\theta \left|\upsilon_H(\theta)^{1/2} \left(\theta^*_H - \theta\right)\right|^2 \le 2$.

Optimality of SMLE θ^*_H for $\theta = (a, b)$, $k = 2$

Theorem 10 (Ben Alaya, N. and Pergamenchtchikov'23)

For any compact set $\Theta \subset]\sigma/2, +\infty[\times]0, +\infty[$ the sequential procedure $\delta^*_H = (\tau^*_H, \theta^*_H)$ is asymptotically optimal in the minimax setting, i.e.

$$\lim_{H\to\infty} \sigma^{-1} H \inf_{(\tau,\widehat{\theta}_\tau)\in\Xi_H} \sup_{\theta\in\Theta} \mathbf{E}_\theta \, (\widehat{\theta}_\tau - \theta)^\top \tilde{F}(\widehat{\theta}_\tau - \theta)$$
$$= \lim_{H\to\infty} \sigma^{-1} H \sup_{\theta\in\Theta} \mathbf{E}_\theta \, (\theta^*_H - \theta)^\top \tilde{F}(\theta^*_H - \theta) = 2.$$

What's new

- The sequential estimation procedures are constructed and non asymptotic mean square accuracy are obtained.

- It should be emphasized, that in the estimation problem for the parameter a, the sequential estimator is well defined and possess the fixed accuracy estimation property in the cases when the classical maximum likelihood estimator is not defined for CIR model.

- Based on the LAN property the minimax estimation theory for the sequential estimation procedures in the continuous time was developed.

- For the first time, the minimax properties for the sequential procedures in the continuous time are obtained in the class of all possible sequential procedures with the same mean observation duration.

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THANK YOU!