Reliable detection of unknown transient change profile by the FMA test

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Plan of presentation

1. Quickest change detection (QCD)
2. Transient change detection (TCD)
3. TCD of known profile by the FMA test. (This part is joint with F. E. Mana, B. K. Guépié, and L. Fillatre)
4. TCD of unknown profile by the FMA test
5. Comparison of the quadratic and linear FMA tests
6. References

The goal of this presentation is twofold :

- ✓ to discuss the passage from the quickest change detection to the reliable transient change detection;
- ✓ to discuss the detection of unknown transient change profile by the FMA test.

Sequential change detection: the very first optimal solutions

Bayesian approach: [Girshick & Rubin 1952, Kolmogorov & Shiryaev 1960, Shiryaev 1961,1963].

Consider the following *continuous time* model of abrupt change :

$$dx_t = \nu \mathbb{1}_{\{t \geq t_0\}} dt + \sigma d\omega_t, \ \ \mathbb{P}_\pi(t_0 < t) = 1 - e^{-\lambda t}$$

where $(\omega_t)_t$ is a normalized Brownian motion. The criterion of the Average Detection Delay (ADD) :

$$\mathrm{ADD}(T) \stackrel{\text{def.}}{=} \mathbb{E}_\pi(T - t_0 | T > t_0) \to \min$$

$$\mathbb{E}_\pi(T | T \leq t_0) \geq \gamma$$

Theorem 1 (Shiryaev 1961)

The optimal solution is given as follows :

$$\mathrm{ADD}(T) = \frac{1}{\rho_{1,0}} [\log \gamma + \log \rho_{1,0} - 1 - C + O(\rho_{1,0})] \ \textit{as} \ \lambda \to 0, \gamma \to \infty, \rho_{1,0} = \frac{\nu^2}{2\sigma^2}.$$

Non bayesian change detection: CUSUM test

[Page 1954]

Consider for some $j : 1 \leq j \leq k$ the hypotheses

$$\begin{aligned} \mathcal{H}_j &= (\xi_1, \ldots, \xi_{j-1}) \sim F_0 \text{ and } (\xi_j, \ldots, \xi_k) \sim F_1 \\ \mathcal{H}_0 &= (\xi_1, \ldots, \xi_k) \sim F_0 \end{aligned}$$

The log-likelihood ratio (LLR) for testing $\mathcal{H}_j$ against $\mathcal{H}_0$ is

$$S_j^k = \log \frac{f_j(\xi_1, \ldots, \xi_k)}{f_0(\xi_1, \ldots, \xi_k)} = \sum_{i=j}^{k} \log \frac{f_1(\xi_i)}{f_0(\xi_i)}$$

Maximum likelihood principle and the recursive form of the CUSUM algorithm :

$$N = \min\{k \geq 1 : g_k \geq h\}, \ g_k = \max_{1 \leq j \leq k} S_j^k = \left(g_{k-1} + \log \frac{f_1(\xi_k)}{f_0(\xi_k)} \right)^+, \ g_0 = 0$$

Non bayesian optimality criterion and an optimal change detection test

[Lorden 1971, Moustakides 1986, Lai 1995, 1998]
Let $\{\xi_k\}_{k\geq 1}$ be (in)dependent random variables (r.v.) observed *sequentially* :

$$\mathcal{L}(\xi_k) = \begin{cases} F_0 & \text{if} \quad k \leq k_0 - 1 \\ F_1 & \text{if} \quad k \geq k_0 \end{cases}, \quad k_0 = 1, 2, \ldots$$

The change time k_0 is an *unknown nonrandom* value. The problem is to *detect* the change in $F_\ell, \ell = 0, 1$ as soon as possible. The criterion is the worst-worst-case mean detection delay :

$$\text{ESADD}(T) \stackrel{\text{def.}}{=} \sup_{k_0 \geq 1} \text{esssup}\, \mathbb{E}_{k_0}((T - k_0 + 1)^+ | \xi_1^{k_0 - 1}) \rightarrow \min$$

over the class $\mathbb{C}_\gamma = \{T : \mathbb{E}_\infty(T) \geq \gamma\}$, where $\mathbb{E}_\infty(T)$ is the Average Run Length (ARL) to false alarm.

Non bayesian optimality criterion and an optimal change detection test

Theorem 2 (Lorden 1971)

If $h = h_\gamma$ is so selected that $\mathbb{E}_\infty(N(h_\gamma)) \geq \gamma$, in particular $h \sim \log \gamma$, then the CUSUM is asymptotically optimal

$$\inf_{T \in \mathbb{C}_\gamma} \mathrm{ESADD}(T) \sim \mathrm{ESADD}(N(h_\gamma)) \sim \frac{\log \gamma}{\rho_{1,0}}, \quad \gamma \to \infty.$$

where $\mathbb{C}_\gamma = \{T : \mathbb{E}_\infty(T) \geq \gamma\}$ *and* $0 < \rho_{1,0} = \mathbb{E}_1 \log \frac{f_1(\xi_i)}{f_0(\xi_i)} < \infty.$

Theorem 3 (Moustakides 1986)

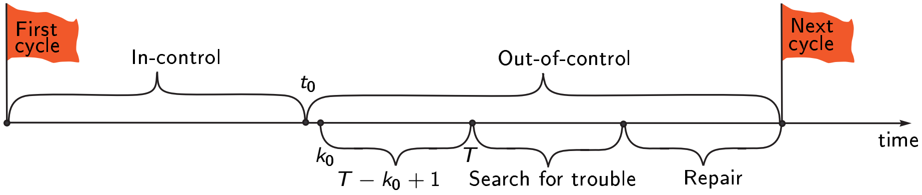
The slightly modified CUSUM test $g_k = (g_{k-1})^+ + \log \frac{f_1(\xi_k)}{f_0(\xi_k)}$ *is optimal for* $\gamma > 1$

$$\inf_{T \in \mathbb{C}_\gamma} \mathrm{ESADD}(T) = \mathrm{ESADD}(N(h)).$$

Motivation for the ARL criterion of QCD : economic criterion

The criterion of the traditional QCD is to minimize the (worst-case) ADD for a given ARL to false alarm. Such a criterion is well adapted to the quality control of the mass-production process. The usage of the ADD and ARL to false alarm is justified by the economic criterion of mass-production process : some runs are short, some other runs are long, but after many repetitions, the optimum is reached (thanks to the CLT).

Maintenance of the life cycle by control charts



Motivation for the ARL criterion of QCD : economic criterion

Economic criterion : [Girshick and Rubin 1952, Duncan 1956, Taylor 1968, Goel and Wu 1973, Chiu 1974,...] The idea of the economic criterion is to minimize the long-run time-average cost (AC) of operation given by

$$\mathrm{AC} = \frac{K_r + (1+\mathbb{E}(N_{\mathrm{f.a.}}))K_s - p\mathbb{E}(t_0) + c\{\triangle t\,[\mathbb{E}(k_0) + \mathrm{ADD} - 1] - \mathbb{E}(t_0)\}}{\mathbb{E}(T_r) + \mathbb{E}(T_s)(1+\mathbb{E}(N_{\mathrm{f.a.}})) + \triangle t\,[\mathbb{E}(k_0) + \mathrm{ADD} - 1]} \to \min,$$

where ADD (or ESADD, or SADD,...) and $\mathbb{E}(N_{\mathrm{f.a.}})$ are functions of the ARL to false alarm : $\mathrm{ADD} \sim \frac{\log \mathrm{ARL}}{\rho_{1,0}}$ and $\mathbb{E}(N_{\mathrm{f.a.}}) = \frac{[\mathbb{E}(k_0)-1]}{\mathrm{ARL}}$

T_s is the time to search for trouble,

T_r is the time for repair,

K_s is the search for trouble cost,

K_r is the repair cost,

p is the profit rate (per hour),

c is the out-of-control cost rate (per hour),

$\triangle t$ is the sampling period.

New twist : the reliable detection of (transient) changes

- ✓ Unlike the traditional QCD, which assumes that the post-change period is infinitely long, sometimes it is necessary to detect a change with an a priori upper bounded detection delay.
- ✓ In such a scenario, all the detections, which exceed the observed phenomena duration or the required time-to-alert L, are assumed missed. Hence, it is natural to define the probability of missed detection as a criterion.
- ✓ To define a class of tests, it is adequate to define the probability of false alarm during a certain reference period.
- ✓ For some safety/security critical systems such as drinking water distribution networks, electric power systems, detecting moving and maneuvering targets or navigation systems integrity monitoring, the main problem is to reliably detect an abrupt change of finite duration.

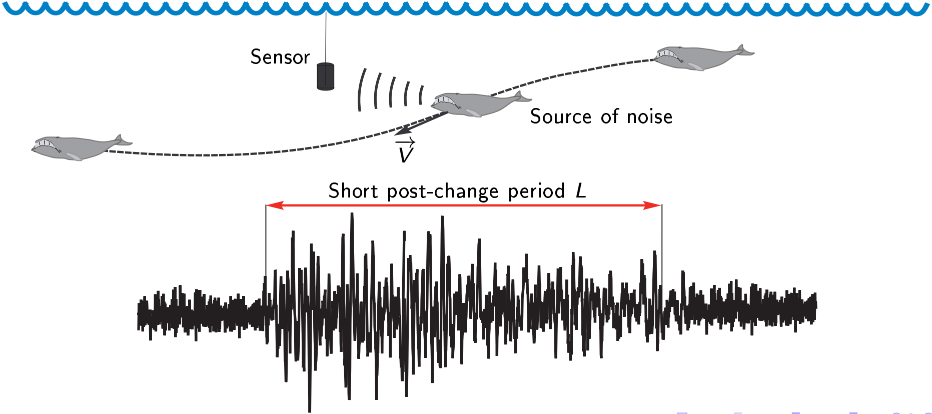
Motivation of the reliable detection of (transient) changes

The reliable detection of (transient) changes is motivated by two possible scenarios

- ✓ The first scenario corresponds to the situation when the observed phenomena is of short and maybe unknown (and random) duration L. Theses changes are often called *transient* (e.g., in underwater acoustics). Sometimes even the "latent" detection (i.e., the detection after the end of transient change) is acceptable. The very first study is
B. Broder and C. Schwartz, "Quickest detection procedures and transient signal detection," Office Naval Res., Arlington, VA, USA, Tech. Rep. 21, 1990.
- ✓ The second scenario arises when the observed anomaly (say, an anomaly in a safety-critical system) leads to a serious degradation of the system safety when the change is detected with the delay greater than the required time-to-alert L, i.e., $T - k_0 + 1 > L$, where T is a stopping time and k_0 is the change-point.

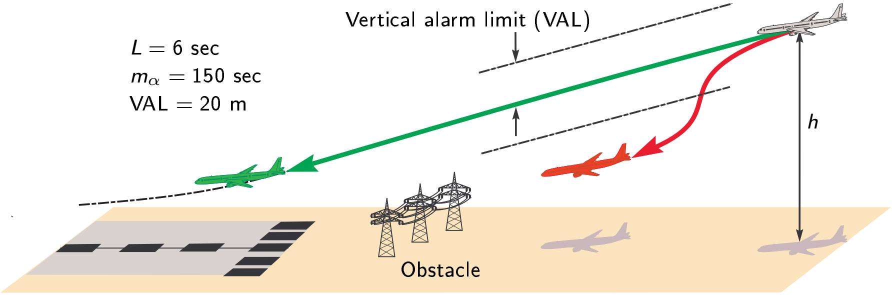
Example of the first scenario : detecting moving and maneuvering objects

The observed phenomena is of short and maybe unknown (and random) duration L, which is defined by the reception/emission conditions, propagation channel, object velocity, etc. Sometimes even the "latent" detection is acceptable.



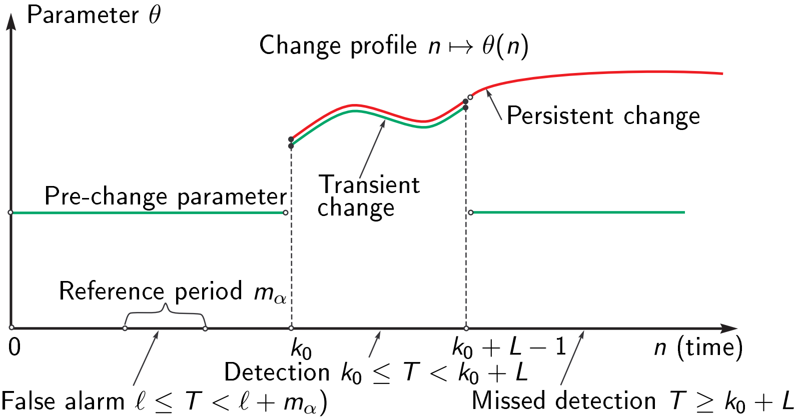
Example of the second scenario : navigation system integrity monitoring

The minimum operational performance for the navigation system (defined by ICAO) specifies the required time-to-alert L, the worst-case missed detection probability and the worst-case probability of false alarm during a given period m_α. The required time-to-alert L is a priori defined by equipment latencies, flight crew reaction time, horizontal/vertical alarm limits, etc; the reference period m_α is equal to the duration of a flight mode.



The reliable detection of (transient) changes

Problem: Calculate the stopping time T which minimizes the probability of missed detection for a given acceptable (!) level of false alarms.



Three classes of transient change detection (TCD) methods :

- ✓ Sequential non-Bayesian approach, the change point is unknown but non random. Sometimes, the duration is unknown : Han et al. (1998, 1999); Streit & Willett (1999); Bakhache & Nikiforov (2000); Wang & Willett (2000); Wang & Willett (2005); Guépié, Fillatre & Nikiforov, (2012, 2017), Moustakides (2014), Noonan & Zhigljavsky (2019, 2020), D. Egea-Roca et al. (2022), Mana, Guépié & Nikiforov (2023), Sokolov, Spivak, & Tartakovsky (2023).
- ✓ Bayesian approach, the appearance, disappearance moments and/or the duration are/is random : Tartakovsky (1987, 1988); Repin (1991); Streit & Willett (1999); Chen & Willett (2000); Trifonov & Korchagin (2001); Premkumar, Kumar, Veeravalli (2010); Tartakovsky et al. (2021);
- ✓ Preliminary transformations of the input data with CUSUM-type or GLR-type algorithms for finite observation intervals : Friedlander & Porat (1989); Broder & Schwartz (1990); Frisch & Messer (1992); Friedlander & Porat (1992); Nuttall (1994, 1996); Stahl & Willett (1997); Streit & Willett (1999); Wang & Willett (2000, 2001).

Criterion of TCD

Let $\{\xi_n\}_{n\geq 1}$ be independent r.v. observed *sequentially*. The generative model of the transient change [Guépié, Fillatre & Nikiforov (2012, 2017)] :

$$\xi_n \sim \begin{cases} F_0 & \text{if} \quad 1 \leq n < k_0 \text{ or } n \geq k_0 + L, \\ F_1^{n-k_0+1} & \text{if} \quad k_0 \leq n \leq k_0 + L - 1 \end{cases}$$

where **the sequence of known (!) CDFs** $F_1^1, \ldots, F_1^L$ defines the profile of transient change.

Criterion :

$$\underset{T \in \mathbb{C}_\alpha}{\text{minimize}} \left\{ \overline{\mathbb{P}}_{\text{md}}(T) = \sup_{k_0 \geq L} \mathbb{P}_{k_0}(T - k_0 + 1 > L \mid T \geq k_0) \right\}$$

among all stopping times $T \in \mathbb{C}_\alpha$ satisfying

$$\mathbb{C}_\alpha = \left\{ T \; : \overline{\mathbb{P}}_{\text{fa}}(T; m_\alpha) = \sup_{\ell \geq L} \mathbb{P}_\infty(\ell \leq T < \ell + m_\alpha) \leq \alpha \right\}$$

Other criteria of TCD

Let $L = 1$. The criterion is [Moustakides (2014)] :

$$\underset{T \in \mathbb{C}_\gamma}{\text{minimize}} \left\{ \overline{\mathbb{P}}_{\text{md}}(T) = \sup \mathbb{P}_{k_0}(T > k_0 \mid T \geq k_0) \right\}$$

among all stopping times $T \in \mathbb{C}_\gamma$ satisfying $\mathbb{C}_\gamma = \{T : \mathbb{E}_\infty T \geq \gamma\}$ The optimal solution is the N-P test $N = \min\{n \geq 1 : \Lambda_n = f_1(\xi_n)/f_0(\xi_n) \geq h\}$.
Let $L \sim \text{Geom}(\varrho)$ be a random geometrically distributed change duration.
The criterion is [Tartakovsky et al. (2021)] :

$$\underset{T \in \widetilde{\mathbb{C}}_\alpha}{\text{minimize}} \left\{ \sup_{k_0 \geq 1} \text{esssup} \, \mathbb{P}_{k_0}(T - k_0 + 1 > L \mid \xi_1^{k_0 - 1}, \; T \geq k_0) \right\}.$$

among all ST $T \in \widetilde{\mathbb{C}}_\alpha = \left\{ T : \sup_{\ell \geq 1} \mathbb{P}_\infty(T \leq \ell + m | T > \ell) \leq \alpha \right\}$. The optimal solution is the modified CUSUM test $T_\varrho = \inf\{n \geq 1 : V_{n,\varrho} \geq h\}$ where $V_{n,\varrho} = \max\{1, V_{n-1,\varrho}\} \Lambda_n (1 - \varrho)$.

Suboptimal solution: WL CUSUM test with variable threshold

Short review of previous results. **(Joint with F. E. Mana, B. K. Guépié, and L. Fillatre)**

The motivation and rationale behind the Window Limited CUSUM test with variable threshold $(h_1, \ldots, h_L)$ is due to the fact that any detection with a delay greater than L is considered missed.

$$T_{\mathrm{WL}}(h) = \inf\left\{n \geq L : \max_{1 \leq k \leq L} \left[S_{n-k+1}^n - h_k\right] \geq 0\right\},$$

$$S_{n-k+1}^n = \sum_{i=n-k+1}^{n} \log \frac{f_{k-n+i}(\xi_i)}{f_0(\xi_i)}.$$

where $f_1, \ldots, f_L$ are the PDF of the distributions $F_1^1, \ldots, F_1^L$.

The WL CUSUM test with variable threshold coincides with

- ✓ the N-P test if $L = 1$;
- ✓ the conventional CUSUM test if $L = \infty$ and $\ell \mapsto h_\ell$ is constant (starting from the first observation $n \geq 1$).

Statistical properties of the WL CUSUM test with variable threshold

Theorem 4 (Guépié, Fillatre & Nikiforov 2017)

1. *The upper bound for the worst-case probability of misdetection* $\overline{\mathbb{P}}_{md}(T_{\mathrm{WL}})$

$$\overline{\mathbb{P}}_{md}(T_{\mathrm{WL}}) \le G(h_L) \stackrel{\mathrm{def.}}{=} \mathbb{P}_{k_0}\left(S_{k_0}^{L+k_0-1} < h_L\right),\ k_0 \ge L.$$

2. *The upper bound for the worst-case probability of false alarm* $\overline{\mathbb{P}}_{fa}(T_{\mathrm{WL}}; m_\alpha)$

$$\overline{\mathbb{P}}_{fa}(T_{\mathrm{WL}}; m_\alpha) \le H(h_1,\dots,h_L) \stackrel{\mathrm{def.}}{=} 1 - \left[\prod_{k=1}^{L} \mathbb{P}_0\left(S_{n-k+1}^{n} < h_k\right)\right]^{m_\alpha}$$

The key point is the assumption that the LLRs $S_L^L,\dots,S_1^L$, $S_{L+1}^{L+1},\dots S_2^{L+1}$, $\dots, S_{L+m_\alpha-1}^{L+m_\alpha-1},\dots,S_{m_\alpha}^{L+m_\alpha-1}$ are associated r.v. under the measure $\mathcal{P}_\infty$.

Definition 1 (Lehmann 1966; Esary, Proschan, & Walkup 1967)

The r.v. $\zeta_1,\dots,\zeta_n$ *are called* **associated** *if* $\mathrm{cov}[f(\zeta_1,\dots,\zeta_n), g(\zeta_1,\dots,\zeta_n)] \ge 0$ *for all coordinatewise nondecreasing functions* f *and* g *for which* $\mathbb{E}[f(\zeta_1,\dots,\zeta_n)]$, $\mathbb{E}[g(\zeta_1,\dots,\zeta_n)]$, *and* $\mathbb{E}[f(\zeta_1,\dots,\zeta_n)g(\zeta_1,\dots,\zeta_n)]$ *exist.*

Optimization of the WL CUSUM test → FMA test

Theorem 5 (Guépié, Fillatre & Nikiforov 2017)

1. *The optimal choice of $h_1,\ldots,h_L$ is reduced to :*

$$\left\{\begin{array}{lcl} \inf\limits_{h_1,\ldots,h_L} \quad G(h_L) & = & \overline{\alpha}_1 \\ \text{subject to} \quad H(h_1,\ldots,h_L) & = & \overline{\alpha}_0 \end{array}\right. .$$

2. *The optimal solution $\{h_i^*, i=1,\ldots,L\}$ of the optimization problem is*

$$h_1^* \to \infty, \ldots, h_{L-1}^* \to \infty, \quad h_L^* = F_{S,L}^{-1}\left((1-\overline{\alpha}_0)^{\frac{1}{m_\alpha}}\right).$$

3. *The smallest value $\overline{\alpha}_1$ of $G(h_L^*)$ as a function of $\overline{\alpha}_0$ is given by*

$$\overline{\alpha}_1 = G\left[F_{S,L}^{-1}\left((1-\overline{\alpha}_0)^{\frac{1}{m_\alpha}}\right)\right].$$

4. *The WL CUSUM test with $\{h_i^*, i=1,\ldots,L\}$ is reduced to the FMA test*

$$T_{FMA} = \inf\left\{n \geq L : S_{n-L+1}^n \geq h_L^*\right\}.$$

Optimization of the WL CUSUM test : arbitrary profile of transient change

The analog of Theorem 5 (assumption on the associated LLRs is relaxed) :

Theorem 6 (Mana, Guepie & Nikiforov 2023)

1. *The optimal solution* $\{h_i^*, i = 1, \dots, L\}$ *of the minimization problem is reached when* $h_1^* \to \infty, \dots, h_{L-1}^* \to \infty$ *and*

$$h_L^* = \begin{cases} F_{S,L}^{-1}\left(1 - \dfrac{\overline{\alpha}_0}{m_\alpha}\right) & \text{if} \quad 1 \leq m_\alpha \leq L \\ F_{S,L}^{-1}\left(1 - \dfrac{m_\alpha - \sqrt{m_\alpha^2 - 4(m_\alpha - L)\overline{\alpha}_0}}{2(m_\alpha - L)}\right) & \text{if} \quad m_\alpha > L \geq 1 \text{ and} \\ & \quad \widetilde{p} \leq \frac{m_\alpha}{2(m_\alpha - L)} \end{cases} .$$

2. *The smallest value* $\overline{\alpha}_1$ *is given by* $\overline{\mathbb{P}}_{md}(T_{FMA}) \leq \overline{\alpha}_1 = G\left(h_L^*(\overline{\alpha}_0)\right)$.

3. *The upper bound for the probability of false alarm of the FMA test is*

$$\overline{\mathbb{P}}_{fa}(T_{FMA}; m_\alpha) \leq \overline{\alpha}_0 = \min\left\{1, m_\alpha \widetilde{p}_o - (m_\alpha - L)^+ \widetilde{p}_o^2\right\},\ \widetilde{p}_o = 1 - F_{S,L}(h_L^*).$$

Unknown profile of transient changes : Gaussian mean case

First attempts to consider this problem [Ph.D. Van Long Do, 2015] and [Guépié, Fillatre & Nikiforov 2017]. Let $\{\xi_n\}_{n\geq 1}$ be independent r.v. observed *sequentially*. Let us consider the generative model of the transient change :

$$\xi_n \sim \begin{cases} \mathcal{N}(0,\sigma^2) & \text{if} \quad 1 \leq n < k_0 \\ \mathcal{N}(\theta_{n-k_0+1},\sigma^2) & \text{if} \quad k_0 \leq n \leq k_0 + L - 1, \end{cases}$$

where the **transient change profile** $\theta = (\theta_1, \ldots, \theta_L)^T$ **is unknown.**
The previously defined WL CUSUM with variable threshold

$$\begin{aligned} T_{\mathrm{WL}}(h) &= \inf\left\{ n \geq L : \max_{1\leq k\leq L} \left[S_{n-k+1}^n - h_k \right] \geq 0 \right\}, \\ S_{n-k+1}^n &= \sum_{i=n-k+1}^{n} \log \frac{f\left(\xi_i, \theta_{k-n+i}^1\right)}{f\left(\xi_i, 0\right)}. \end{aligned}$$

where $h = (h_1, \ldots, h_L)$ and $f(x,\theta_1^1), \ldots, f(x,\theta_L^1)$ are the PDF corresponding to the profile of distributions $\mathcal{N}(\theta_1^1,\sigma^2), \ldots, \mathcal{N}(\theta_L^1,\sigma^2)$, necessitates the definition of the putative profile $\theta_1 = (\theta_1^1, \ldots, \theta_L^1)^T$, which can be seriously different from the true one θ.

Unknown profile of transient changes : Gaussian mean case

The stopping time of the linear FMA (LFMA) test with the putative profile θ_1 is given as follows

$$T_{\text{LFMA}} = \inf\left\{ n \geq L : S_{n-L+1}^{n} = \frac{1}{\sigma^2} \sum_{i=n-L+1}^{n} \theta_{L-n+i}^{1} \xi_i \geq h_L^* \right\}.$$

What happens if the putative profile θ_1 is different from the true profile θ of the previously defined generative model ?

The smallest value $\overline{\alpha}_1$ of $G(h_L^*)$ provided that the upper bound for $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{LFMA}}; m_\alpha)$ is equal to a pre-assigned value $\overline{\alpha}_0$, is given by

$$\overline{\mathbb{P}}_{\text{md}}(T_{\text{LFMA}}) \leq \overline{\alpha}_1 = G(h_L^*) = \Phi\left(\frac{\sigma h_L^*}{\|\theta_1\|_2} - \frac{\langle \theta \cdot \theta_1 \rangle}{\sigma \, \|\theta_1\|_2} \right),$$

where $\langle x \cdot y \rangle = \sum_{i=1}^{L} x_i y_i$ is the inner product of two vectors x and y.

Unknown profile of transient changes : generalized likelihood ratio

Since the profile $\theta = (\theta_1, \ldots, \theta_L)^T$ is unknown, it is proposed to estimate the unknown parameters of the PDF $\theta_1, \ldots, \theta_k$, $1 \leq k \leq L$, by using the vector of observations $\xi = (\xi_{n-k+1}, \ldots, \xi_n)^T$:

$$\widehat{\theta} = \arg\max_{\theta \in \mathbb{R}^k} f(\xi_{n-k+1}, \ldots, \xi_n; \theta_1, \ldots, \theta_k), \quad \theta = (\theta_1, \ldots, \theta_k)^T$$

The GLR $\widehat{S}_{n-k+1}^n = 2 \log \widehat{\Lambda}(\xi_{n-k+1}, \ldots, \xi_n)$ for testing between the hypotheses $\mathcal{H}_0 = \{\theta = 0\}$ and $\mathcal{H}_1 = \{\theta \neq 0\}$ is defined as follows :

$$\widehat{S}_{n-k+1}^n = 2 \log \frac{\max_{\theta \in \mathbb{R}^k} f(\xi_{n-k+1}, \ldots, \xi_n; \theta)}{f(\xi_{n-k+1}, \ldots, \xi_n; \theta = 0)}$$

$$= 2 \log \frac{\max\limits_{\theta \in \mathbb{R}^k} \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=n-k+1}^{n} (\xi_i - \theta_{k-n+i})^2\right\}}{\exp\left\{-\frac{1}{2\sigma^2} \sum_{i=n-k+1}^{n} \xi_i^2\right\}} = \frac{1}{\sigma^2} \|\xi\|_2^2.$$

Unknown profile of transient changes : generalized likelihood ratio

The WL CUSUM with variable threshold based on the GLR

$$T_{\mathrm{WL}}(h) = \inf \left\{ n \geq L : \max_{1 \leq k \leq L} \left[\widehat{S}_{n-k+1}^{n} - h_k \right] \geq 0 \right\},$$

$$\widehat{S}_{n-k+1}^{n} = \frac{1}{\sigma^2} \sum_{i=n-k+1}^{n} \xi_i^2 .$$

The GLR $\widehat{S}_L^L, \ldots, \widehat{S}_1^L, \widehat{S}_{L+1}^{L+1}, \ldots \widehat{S}_2^{L+1}, \ldots, \widehat{S}_{L+m_\alpha-1}^{L+m_\alpha-1}, \ldots, \widehat{S}_{m_\alpha}^{L+m_\alpha-1}$ are associated under the measure $\mathcal{P}_\infty$. By using Theorems 4 and 5, we get the following :

Theorem 7

1. *The optimal solution* $\{h_i^*, i = 1, \ldots, L\}$ *of the optimization problem is reached when* $h_1 \to \infty, \ldots, h_{L-1} \to \infty$ *and*

$$h_L^* = F_{S,L}^{-1}\left((1 - \overline{\alpha}_0)^{\frac{1}{m_\alpha}} \right) = F_{\chi^2}^{-1}\left((1 - \overline{\alpha}_0)^{\frac{1}{m_\alpha}} ; L \right),$$

where $x \mapsto F_{\chi^2}^{-1}(x; L)$ *is the quantile function of the* χ^2 *distribution with* L *degrees of freedom.*

Unknown profile of transient changes : operating characteristics

2. *The smallest value* $\overline{\alpha}_1$ *of* $G(h_L^*)$ *provided that the upper bound for* $\overline{\mathbb{P}}_{fa}(T_{QFMA}; m_\alpha)$ *is equal to a pre-assigned value* $\overline{\alpha}_0$*, is given by*

$$\overline{\alpha}_1 = G\left[F_{S,L}^{-1}\left((1-\overline{\alpha}_0)^{\frac{1}{m_\alpha}}\right)\right] = F_{\chi^2}\left[F_{\chi^2}^{-1}\left((1-\overline{\alpha}_0)^{\frac{1}{m_\alpha}}; L\right); L, \lambda\right],$$

where $x \mapsto F_{\chi^2}(x; L, \lambda)$ *is the CDF of the noncentral* χ^2 *distribution with* L *degrees of freedom and noncentrality parameter* $\lambda = \frac{1}{\sigma^2}\|(\theta_1, \ldots, \theta_L)\|_2^2$.

3. *The stopping time of the optimized WL CUSUM test with variable threshold is reduced to the stopping time* T_{QFMA} *of the QFMA test*

$$T_{QFMA} = \inf\left\{n \geq L : \widehat{S}_{n-L+1}^n = \frac{1}{\sigma^2}\sum_{i=n-L+1}^{n} \xi_i^2 \geq h_L^*\right\}.$$

Unknown profiles before and after transient changes

Let $\{\xi_n\}_{n\geq 1}$ be independent Gaussian r.v. observed *sequentially*, $\xi_n \sim \mathcal{N}(\theta_n, \sigma^2)$. A more general situation is considered now : the mean θ_n is **unknown before and after change**. The inequality constraints are imposed on the norms $\|\ldots\|_2$ of the profile vector θ before and after change.

Let us consider the following extended generative model of the transient change :

$$\xi_n \sim \begin{cases} \mathcal{N}(\theta_n, \sigma^2), & \text{if} \quad 1 \leq n < k_0 \\ \mathcal{N}(\theta_{n-k_0+1}, \sigma^2) & \text{if} \quad k_0 \leq n \leq k_0 + L - 1, \end{cases}$$

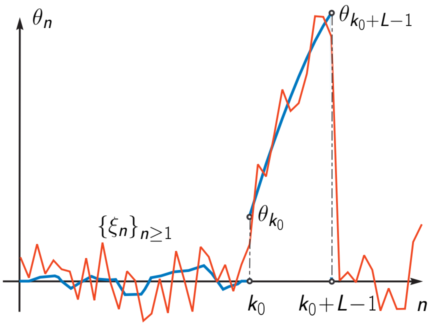
where the constraints on the norms of the vector $\theta = (\theta_{n-L+1}, \ldots, \theta_n)^T$ are defined as follows :

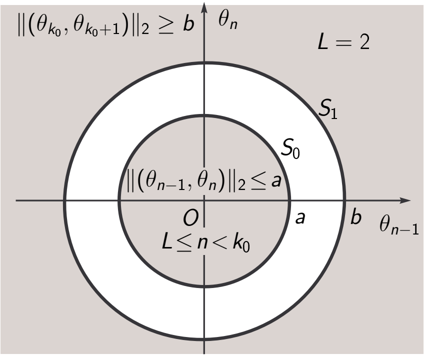
$$\begin{array}{lll} \|(\theta_{n-L+1}, \ldots, \theta_n)\|_2^2 \leq a^2 & \text{if} & L \leq n < k_0 \\ \|(\theta_{n-L+1}, \ldots, \theta_n)\|_2^2 \geq b^2 & \text{if} & n = k_0 + L - 1. \end{array}$$

Unknown profiles before and after transient changes

The parametric regions of θ before and after transient changes are limited by two concentric spherical surfaces S_0 with radius a and S_1 with radius b defined by the following equations

$$S_0 \; : \|(\theta_{n-L+1},\ldots,\theta_n)\|_2 = a, \quad S_1 \; : \|(\theta_{n-L+1},\ldots,\theta_n)\|_2 = b.$$





Unknown profiles before and after transient changes : GLR

Let us consider the problem of testing between the following hypotheses :

$$\mathcal{H}_0 = \{\theta : \|\theta\|_2 \leq a\} \text{ and } \mathcal{H}_1 = \{\theta : \|\theta\|_2 \geq b\},$$

where $0 \leq a < b < \infty$, by using the observations $\xi_{n-L+1}, \ldots, \xi_n$, $\xi_i \sim \mathcal{N}(\theta_{L-n+i}, \sigma^2)$. The GLR can be written as :

$$\begin{aligned}
\widehat{S}^n_{n-L+1} &= 2 \log \hat{\Lambda}(\xi_{n-L+1}, \ldots, \xi_n) \\
&= 2 \log \max_{\|\theta\|_2 \geq b} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=n-L+1}^{n} (\xi_i - \theta_{L-n+i})^2 \right\} \\
&\quad -2 \log \max_{\|\theta\|_2 \leq a} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=n-L+1}^{n} (\xi_i - \theta_{L-n+i})^2 \right\}.
\end{aligned}$$

Unknown profiles before and after transient changes : GLR

After simple transformations :

$$\widehat{S}_{n-L+1}^{n} = \max_{\|\theta\|_2 \geq b} \left\{ -\frac{1}{\sigma^2} \|\theta - \xi\|_2^2 \right\} - \max_{\|\theta\|_2 \leq a} \left\{ -\frac{1}{\sigma^2} \|\theta - \xi\|_2^2 , \right\}$$

where $\xi = (\xi_{n-L+1}, \ldots, \xi_n)^T$ and $\theta = (\theta_1, \ldots, \theta_L)^T$.
This equation can be re-written as [Borovkov 1998] :

$$\widehat{S}_{n-L+1}^{n} = \begin{cases} -\frac{1}{\sigma^2}(\|\xi\|_2 - b)^2 & \text{if} \quad \|\xi\|_2 \leq a \\ -\frac{1}{\sigma^2}(\|\xi\|_2 - b)^2 + \frac{1}{\sigma^2}(\|\xi\|_2 - a)^2 & \text{if} \quad a \leq \|\xi\|_2 \leq b \\ +\frac{1}{\sigma^2}(\|\xi\|_2 - a)^2 & \text{if} \quad \|\xi\|_2 \geq b \end{cases} .$$

The GLR $\widehat{S}_{n-L+1}^{n}$ is a continuous increasing function of $\|\xi\|_2$.

Unknown profiles before and after transient changes : operating characteristics

Corollary 1

1. *The threshold h_L^* of the QFMA test is calculated now as follows*

$$h_L^* = F_{S,L}^{-1}\left((1-\overline{\alpha}_0)^{\frac{1}{m_\alpha}}\right) = F_{\chi^2}^{-1}\left((1-\overline{\alpha}_0)^{\frac{1}{m_\alpha}}; L, \frac{a^2}{\sigma^2}\right),$$

where $x \mapsto F_{\chi^2}^{-1}(x; L, \lambda)$ is the quantile function of the noncentral χ^2 distribution with L degrees of freedom and noncentrality parameter $\lambda = \frac{a^2}{\sigma^2}$.

2. *The smallest value $\overline{\alpha}_1$ of $G(h_L^*)$ provided that the upper bound for $\overline{\mathbb{P}}_{fa}(T_{QFMA}; m_\alpha)$ is equal to a pre-assigned value $\overline{\alpha}_0$, is given by*

$$\overline{\alpha}_1 = G\left[F_{S,L}^{-1}\left((1-\overline{\alpha}_0)^{\frac{1}{m_\alpha}}\right)\right] = F_{\chi^2}\left[F_{\chi^2}^{-1}\left((1-\overline{\alpha}_0)^{\frac{1}{m_\alpha}}; L, \frac{a^2}{\sigma^2}\right); L, \frac{b^2}{\sigma^2}\right],$$

where $x \mapsto F_{\chi^2}(x; L, \lambda)$ is the CDF of the noncentral χ^2 distribution with L degrees of freedom and noncentrality parameter $\lambda = \frac{b^2}{\sigma^2}$.

Unknown profile of transient changes : linear model with nuisance parameters

Let us consider the following linear Gaussian model with transient changes; the case of persistent change in [Fouladirad & Nikiforov 2005, 2006]

$$Y_n = HX_n + M\widetilde{\theta}_n + \xi_n, \;\; \widetilde{\theta}_n = \begin{cases} 0 & \text{if} \quad 1 \leq n < k_0 \\ \theta_{n-k_0+1} & \text{if} \quad k_0 \leq n \leq k_0 + L - 1, \end{cases},$$

where $X_n \in \mathbb{R}^m$ is an unknown and non-random **nuisance parameter**, M is a full column rank matrix of size $(\ell \times r)$ with $r < \ell$ and H is a matrix of size $(\ell \times m)$ with $\mathrm{rank} H = q$, $\xi_n \sim \mathcal{N}(0, \sigma^2 I_\ell)$.

The problem remains invariant under the group of translations $G = \{g : g(Y) = Y + HC\}$, $C \in \mathbb{R}^m$. The solution is the projection of Y_n on the orthogonal complement $R(H)^\perp$ of the column space $R(H)$:

$$Z_n = WY_n, \quad WH = 0, \;\; W^T W = P_H, \;\; WW^T = I_{\ell-q}.$$

where $W^T = (w_1, \ldots, w_{\ell-q})$ of size $\ell \times (\ell - q)$ is composed of the eigenvectors $w_1, \ldots, w_{\ell-q}$ of $P_H = I_\ell - H(H^T H)^- H^T$ corresponding to eigenvalues 1. If $q = m$, then $P_H = I_\ell - H(H^T H)^{-1} H^T$.

Unknown profile of transient changes : linear model with nuisance parameters

Let WM be full column rank of size $(\ell - q) \times r$ and $r \leq \ell - q$.
The generative model in $R(H)^{\perp}$ is

$$Z_n = WM\widetilde{\theta}_n + \zeta_n, \quad \widetilde{\theta}_n = \begin{cases} 0 & \text{if} \quad 1 \leq n < k_0 \\ \theta_{n-k_0+1} & \text{if} \quad k_0 \leq n \leq k_0 + L - 1, \end{cases},$$

where $\zeta_n \sim \mathcal{N}(0, \sigma^2 I_{\ell-q})$ and $\widetilde{\theta}_n, \theta_1, \ldots, \theta_L \in \mathbb{R}^r$. The GLR is written as :

$$\widehat{S}_{n-L+1}^{n} = \frac{1}{\sigma^2} Z^T A Z, \quad A = \mathrm{diag}\left\{B, B, \ldots, B\right\} \quad Z = (Z_{n-L+1}^T, \ldots, Z_n^T)^T,$$

where $B = (WM)\left[(WM)^T(WM)\right]^{-1}(WM)^T$ is idempotent and symmetric of rank r. The stopping time T_{QFMA} of the QFMA test

$$T_{\text{QFMA}} = \inf\left\{n \geq L : \widehat{S}_{n-L+1}^{n} = \frac{1}{\sigma^2} Z^T A\, Z \geq h_L^*\right\}.$$

Unknown profile of transient changes : linear model with nuisance parameters

Corollary 2

1. *The GLR* $\widehat{S}_L^L, \ldots, \widehat{S}_1^L$, $\widehat{S}_{L+1}^{L+1}, \ldots \widehat{S}_2^{L+1}, \ldots, \widehat{S}_{L+m_\alpha-1}^{L+m_\alpha-1}, \ldots, \widehat{S}_{m_\alpha}^{L+m_\alpha-1}$ *are associated under the measure* $\mathcal{P}_\infty$ *and Theorem 5 is used. The GLR* $\widehat{S}_{n-L+1}^n$ *follows the* χ^2 *distribution with* $L{\cdot}r$ *degrees of freedom.*
2. *The threshold of the QFMA test*

$$h_L^* = F_{\chi^2}^{-1}\left((1-\overline{\alpha}_0)^{\frac{1}{m_\alpha}}; L{\cdot}r\right),$$

where $x \mapsto F_{\chi^2}^{-1}(x; L{\cdot}r)$ *is the quantile function of the* χ^2 *distribution with* $L{\cdot}r$ *degrees of freedom and* $\overline{\alpha}_0$ *is the pre-assigned value of the upper bound for* $\overline{\mathbb{P}}_{fa}(T_{QFMA}; m_\alpha)$.

3. *The upper bound for* $\overline{\mathbb{P}}_{md}(T_{QFMA})$ *is*

$$\overline{\mathbb{P}}_{md}(T_{QFMA}) \leq \overline{\alpha}_1 = F_{\chi^2}(h_L^*; L{\cdot}r, \lambda) \text{ with } \lambda = \frac{1}{\sigma^2}\sum_{i=1}^{L} \theta_i^T M^T P_H M \theta_i.$$

Comparison of two QFMA tests with the LFMA test

Let us consider the following FMA tests :

✓ The linear FMA (LFMA) test with the putative profile θ_1

$$T_{\text{LFMA}} = \inf \left\{ n \geq L : S_{n-L+1}^{n} = \frac{1}{\sigma^2} \sum_{i=n-L+1}^{n} \theta_{L-n+i}^{1} \xi_i \geq h_L^* \right\}.$$

✓ The quadratic FMA (QFMA_C) test with unknown constant profile

$$T_{\text{QFMA}_C} = \inf \left\{ n \geq L : \widehat{S}_{n-L+1}^{n} = \frac{1}{\sigma^2} \left(\sum_{i=n-L+1}^{n} \xi_i \right)^2 \geq h_L^* \right\}.$$

✓ The quadratic FMA (QFMA_D) test with unknown dynamic profile

$$T_{\text{QFMA}_D} = \inf \left\{ n \geq L : \widehat{S}_{n-L+1}^{n} = \frac{1}{\sigma^2} \sum_{i=n-L+1}^{n} \xi_i^2 \geq h_L^* \right\}.$$

QFMA$_\text{D}$ test (unknown profile) vs. LFMA test (putative profile)

The upper bounds for the probability of missed detection of the LFMA and QFMA$_\text{D}$ tests as functions of $\overline{\alpha}_0$ are given as follows

$$\overline{\mathbb{P}}_{\text{md}}(T_{\text{LFMA}}) \leq \overline{\alpha}_1 = \begin{cases} \Phi\left(\Phi^{-1}\Big((1-\overline{\alpha}_0)^{\frac{1}{m_\alpha}}\Big) - \frac{\langle\theta\cdot\theta_1\rangle}{\sigma\|\theta_1\|_2}\right) & \text{for constant sign } \theta_1 \\ \Phi\left(\Phi^{-1}\Big(1-\frac{\overline{\alpha}_0}{m_\alpha}\Big) - \frac{\langle\theta\cdot\theta_1\rangle}{\sigma\|\theta_1\|_2}\right) & \text{for arbitrary } \theta_1 \end{cases},$$

where $x \mapsto \Phi^{-1}(x)$ is the quantile function of the standard normal distribution, and

$$\overline{\mathbb{P}}_{\text{md}}(T_{\text{QFMA}_D}) \leq \overline{\alpha}_1 = F_{\chi^2}\left[F_{\chi^2}^{-1}\left((1-\overline{\alpha}_0)^{\frac{1}{m_\alpha}}; L\right); L, \frac{\|\theta\|_2^2}{\sigma^2}\right],$$

where $x \mapsto F_{\chi^2}^{-1}(x; L, \lambda)$ is the quantile function of the noncentral χ^2 distribution with L degrees of freedom and noncentrality parameter $\frac{\|\theta\|_2^2}{\sigma^2}$.

QFMA$_D$ test (unknown profile) vs. LFMA test (putative profile)

Let us define the frontier between two subsets of advantages for the LFMA and QFMA$_D$ tests from the condition of equal power

$$\overline{\alpha}_1(T_{\mathrm{LFMA}}) = \overline{\alpha}_1(T_{\mathrm{QFMA}_D}).$$

The cosine f of the angle $\beta = \angle(\theta, \theta_f)$ between the true dynamic profile θ and the profile θ_f, calculated from the above mentioned equal power condition, defines such a frontier :

$$\cos(\angle(\theta, \theta_1)) = \frac{\langle \theta \cdot \theta_1 \rangle}{\|\theta\|_2 \cdot \|\theta_1\|_2} \gtrless \cos(\beta) = f\left(\overline{\alpha}_0, L, m_\alpha, \frac{\|\theta\|_2^2}{\sigma^2}\right)$$

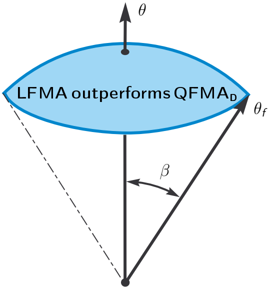
where

$$f = \frac{\sigma}{\|\theta\|_2}\left\{\Phi^{-1}\left(1 - \frac{\overline{\alpha}_0}{m_\alpha}\right) - \Phi^{-1}\left(F_{\chi^2}\left[F_{\chi^2}^{-1}\left((1-\overline{\alpha}_0)^{\frac{1}{m_\alpha}}; L\right); L, \frac{\|\theta\|_2^2}{\sigma^2}\right]\right)\right\}$$

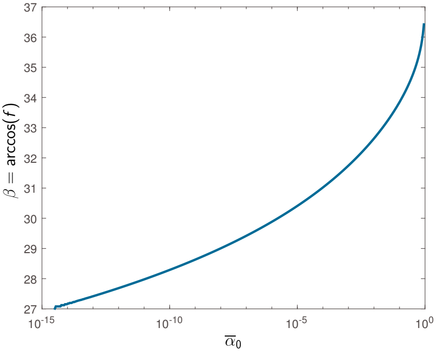
$QFMA_D$ test (unknown profile) vs. LFMA test (putative profile)

Let $L = 6$, $m_\alpha = 20$, the true profile $\theta = (2,\ -5,\ 1,\ 2,\ 3,\ 4)^T$, $\sigma^2 = 1$.

The solid angle of a cone with apex angle $2\beta = 2\arccos(f)$ and axis θ defines the subset of θ_1, where LFMA outperforms $QFMA_D$.



The angle $\beta = \arccos(f)$ as a function of the upper bound $\overline{\alpha}_0$ for the probability of false alarm.



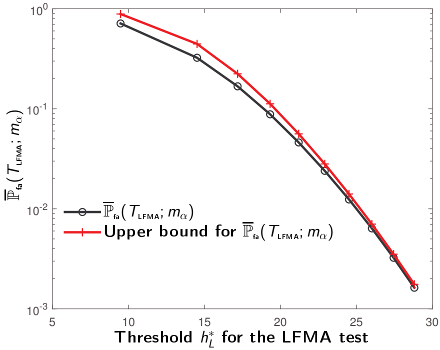
QFMA_D test (unknown profile) vs. LFMA test (putative profile)

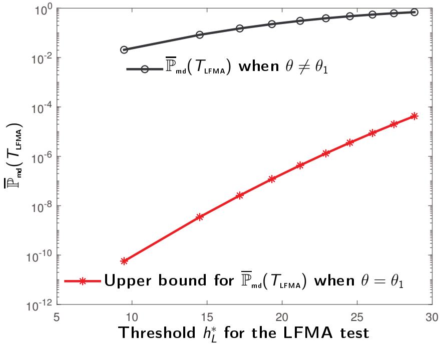
Let $L=6$, $m_\alpha=20$, the true profile $\theta=(2, \mathbf{-5}, 1, 2, 3, 4)^T$, the putative profile $\theta_1=(2, \mathbf{5}, 1, 2, 3, 4)^T$, and $\sigma^2=1$. **Hence,** $\angle(\theta, \theta_1)=81.2^\circ > \beta$.

✓ Upper bounds for $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{LFMA}}; m_\alpha)$ and $\overline{\mathbb{P}}_{\text{md}}(T_{\text{LFMA}})$.

✓ $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{LFMA}}; m_\alpha)$ and $\overline{\mathbb{P}}_{\text{md}}(T_{\text{LFMA}})$ by numerical calculation.

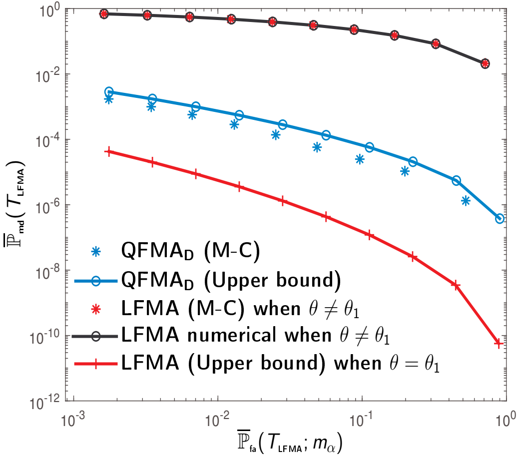
✓ LFMA test with putative dynamic profile θ_1 of transient change.





QFMA_D test (unknown profile) vs. LFMA test (putative profile)

Let $L = 6$, $m_\alpha = 20$, the true profile $\theta = (2, \mathbf{-5}, 1, 2, 3, 4)^T$, the putative profile $\theta_1 = (2, \mathbf{5}, 1, 2, 3, 4)^T$, and $\sigma^2 = 1$. **Hence, $\angle(\theta, \theta_1) = 81.2^\circ > \beta$.**



- ✓ Upper bounds for $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{LFMA}}; m_\alpha)$, $\overline{\mathbb{P}}_{\text{md}}(T_{\text{LFMA}})$.
- ✓ Upper bounds for $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{QFMA}_D}; m_\alpha)$, $\overline{\mathbb{P}}_{\text{md}}(T_{\text{QFMA}_D})$.
- ✓ $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{LFMA}}; m_\alpha)$, $\overline{\mathbb{P}}_{\text{md}}(T_{\text{LFMA}})$ by numerical calculation.
- ✓ $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{LFMA}}; m_\alpha)$, $\overline{\mathbb{P}}_{\text{md}}(T_{\text{LFMA}})$, $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{QFMA}_D}; m_\alpha)$, $\overline{\mathbb{P}}_{\text{md}}(T_{\text{QFMA}_D})$ by M-C simulation.
- ✓ LFMA test with putative dynamic profile θ_1 of transient change.

Two QFMA tests : unknown dynamic profile vs. unknown constant profile

The upper bounds for the probability of missed detection of the QFMA_C and QFMA_D tests as functions of $\overline{\alpha}_0$ are given as follows

$$\overline{\mathbb{P}}_{\text{md}}(T_{\text{QFMA}_C}) \leq \overline{\alpha}_1 = F_{\chi^2}\left[F^{-1}_{\chi^2}\left(1-\frac{\overline{\alpha}_0}{m_\alpha};1\right);1,\cos^2(\angle(\mathbb{1},\theta))\frac{\|\theta\|_2^2}{\sigma^2}\right],$$

where $x \mapsto F^{-1}_{\chi^2}(x;1,\lambda)$ is the quantile function of the noncentral χ^2 distribution with 1 degree of freedom and noncentrality parameter $\cos^2(\angle(\mathbb{1},\theta))\frac{\|\theta\|_2^2}{\sigma^2}$, $\mathbb{1}=(1,1,\ldots,1)^T$, and

$$\overline{\mathbb{P}}_{\text{md}}(T_{\text{QFMA}_D}) \leq \overline{\alpha}_1 = F_{\chi^2}\left[F^{-1}_{\chi^2}\left((1-\overline{\alpha}_0)^{\frac{1}{m_\alpha}};L\right);L,\frac{\|\theta\|_2^2}{\sigma^2}\right],$$

where $x \mapsto F^{-1}_{\chi^2}(x;L,\lambda)$ is the quantile function of the noncentral χ^2 distribution with L degrees of freedom and noncentrality parameter $\frac{\|\theta\|_2^2}{\sigma^2}$.

Two QFMA tests : unknown dynamic profile vs. unknown constant profile

Let us define the frontiers between the subsets of advantages for the QFMA_{C} and QFMA_{D} tests from the condition of equal power

$$\overline{\alpha}_1(T_{\text{QFMA}_C}) = \overline{\alpha}_1(T_{\text{QFMA}_D}).$$

The trajectory of the vector θ_f defines such a frontier. The cosine f of the angle $\beta = \angle(\mathbb{1}, \theta_f)$ between the vector $\mathbb{1} = (1, 1, \ldots, 1)^T$ and the dynamic profile θ_f is calculated from the above mentioned equal power condition by numerical solving the following equation :

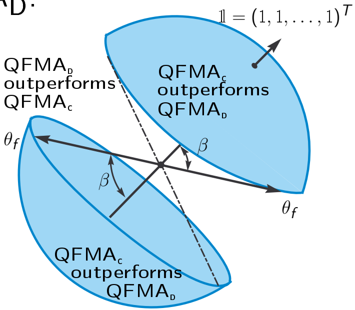
$$F_{\chi^2}\left[F_{\chi^2}^{-1}\left(1 - \frac{\overline{\alpha}_0}{m_\alpha}; 1\right); 1, \cos^2(\beta)\frac{\|\theta_f\|_2^2}{\sigma^2}\right] = F_{\chi^2}\left[F_{\chi^2}^{-1}\left((1-\overline{\alpha}_0)^{\frac{1}{m_\alpha}}; L\right); L, \frac{\|\theta_f\|_2^2}{\sigma^2}\right]$$

$$|\cos(\angle(\mathbb{1}, \theta))| = \frac{|\langle \mathbb{1} \cdot \theta \rangle|}{\sqrt{L} \cdot \|\theta\|_2} \gtrless |\cos(\beta)| = f\left(\overline{\alpha}_0, L, m_\alpha, \frac{\|\theta_f\|_2^2}{\sigma^2}\right).$$

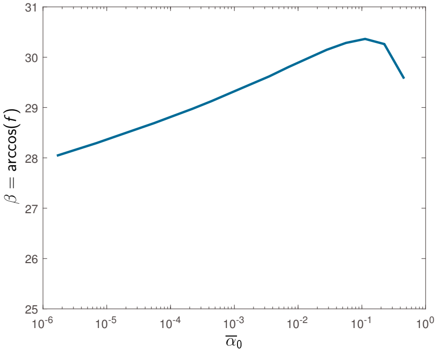
Two QFMA tests : unknown dynamic profile vs. unknown constant profile

Let $L = 6$, $m_\alpha = 20$, the true profile $\theta = (5, 2, 1, 2, 3, 4)^T$, $\sigma^2 = 1$.

The solid angles of a double cone with apex angle $2\beta = 2\arccos(f)$ and axis $\mathbb{1} = (1, 1, \ldots, 1)^T$ define the subsets of θ, where QFMA_C outperforms QFMA_D.



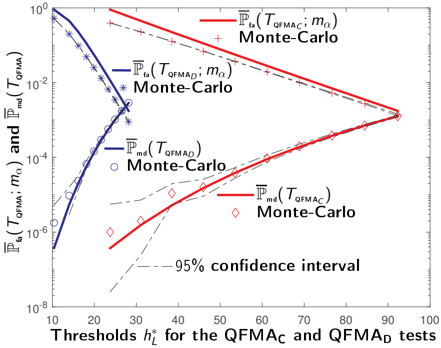
The angle $\beta = \arccos(f)$ as a function of the upper bound $\overline{\alpha}_0$ for the probability of false alarm.

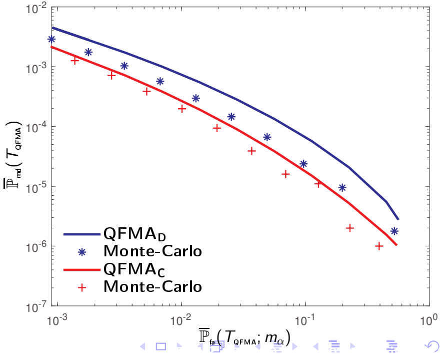


Two QFMA tests : unknown dynamic profile vs. unknown constant profile

Let $L=6$, $m_\alpha=20$, $\theta=(5,\ 2,\ 1,\ 2,\ 3,\ 4)^T$, $\sigma^2=1$, $\angle(\mathbb{1},\theta)=25.4^\circ<\beta(\overline{\alpha}_0)$.

- ✓ Upper bounds for $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{QFMA}_C};m_\alpha)$, $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{QFMA}_D};m_\alpha)$, $\overline{\mathbb{P}}_{\text{md}}(T_{\text{QFMA}_C})$ and $\overline{\mathbb{P}}_{\text{md}}(T_{\text{QFMA}_D})$.
- ✓ $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{QFMA}_C};m_\alpha)$, $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{QFMA}_D};m_\alpha)$, $\overline{\mathbb{P}}_{\text{md}}(T_{\text{QFMA}_C})$ and $\overline{\mathbb{P}}_{\text{md}}(T_{\text{QFMA}_D})$ by 10^6-repetition M-C simulation.
- ✓ QFMA_D = unknown dynamic profile; QFMA_C =unknown constant profile.

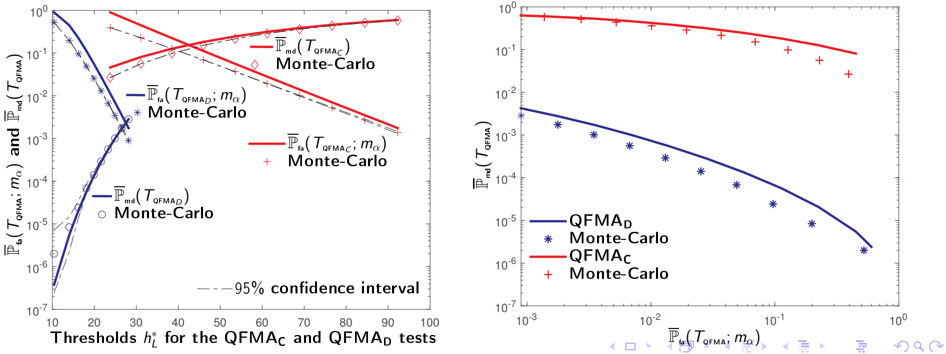




Two QFMA tests : unknown dynamic profile vs. unknown constant profile

Let $L=6$, $m_\alpha=20$, $\theta=(5,-2,1,-2,3,4)^T$, $\sigma^2=1$, $\angle(\mathbb{1},\theta)=61.4^\circ>\beta(\overline{\alpha}_0)$.

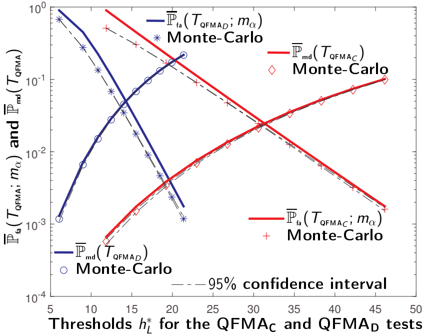
- ✓ Upper bounds for $\overline{\mathbb{P}}_{\mathrm{fa}}(T_{\mathrm{QFMA}_C};m_\alpha)$, $\overline{\mathbb{P}}_{\mathrm{fa}}(T_{\mathrm{QFMA}_D};m_\alpha)$, $\overline{\mathbb{P}}_{\mathrm{md}}(T_{\mathrm{QFMA}_C})$ and $\overline{\mathbb{P}}_{\mathrm{md}}(T_{\mathrm{QFMA}_D})$.
- ✓ $\overline{\mathbb{P}}_{\mathrm{fa}}(T_{\mathrm{QFMA}_C};m_\alpha)$, $\overline{\mathbb{P}}_{\mathrm{fa}}(T_{\mathrm{QFMA}_D};m_\alpha)$, $\overline{\mathbb{P}}_{\mathrm{md}}(T_{\mathrm{QFMA}_C})$ and $\overline{\mathbb{P}}_{\mathrm{md}}(T_{\mathrm{QFMA}_D})$ by 10^6-repetition M-C simulation.
- ✓ $\mathrm{QFMA_D}$ = unknown dynamic profile; $\mathrm{QFMA_C}$ =unknown constant profile.

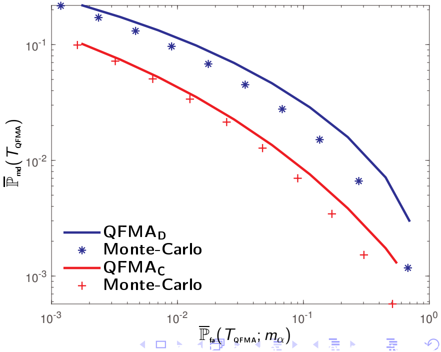


Two QFMA tests : unknown dynamic profile vs. unknown constant profile

Let $L = 3,\ m_\alpha = 20,\ \theta = (3,3,3)^T,\ \sigma^2 = 1,\ \angle(\mathbb{1},\theta) = 0^\circ < \beta(\overline{\alpha}_0)$.

- ✓ Upper bounds for $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{QFMA}_C}; m_\alpha)$, $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{QFMA}_D}; m_\alpha)$, $\overline{\mathbb{P}}_{\text{md}}(T_{\text{QFMA}_C})$ and $\overline{\mathbb{P}}_{\text{md}}(T_{\text{QFMA}_D})$.
- ✓ $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{QFMA}_C}; m_\alpha)$, $\overline{\mathbb{P}}_{\text{fa}}(T_{\text{QFMA}_D}; m_\alpha)$, $\overline{\mathbb{P}}_{\text{md}}(T_{\text{QFMA}_C})$ and $\overline{\mathbb{P}}_{\text{md}}(T_{\text{QFMA}_D})$ by 10^6-repetition M-C simulation.
- ✓ QFMA_D = unknown dynamic profile; QFMA_C =unknown constant profile.





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