Multilevel Monte Carlo for pricing Barrier options under CIR, CEV and Heston models

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Outline of The Talk

1. The one-dimensional setting with locally Lipschitz diffusion coefficient

2. The two-dimensional setting : Heston model

Outline

Motivation

- Down-and-Out (D-O) and the Up-and-Out (U-O) barrier options

$$\pi_{\mathcal{B}_D} = \mathbb{E}\left[f(X_T)1_{\{\inf_{t\in[0,T]} X_t > \mathcal{B}_D\}}\right] \text{ and } \pi_{\mathcal{B}_U} = \mathbb{E}\left[f(X_T)1_{\{\sup_{t\in[0,T]} X_t < \mathcal{B}_U\}}\right]$$

for

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x,$$

where $(W_t)_{t\geq 0}$ is a s.B.M. $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}_+^*$ are loc. Lipschitz-functions such that $\frac{1}{\sigma}$ is loc. integrable.

- For $\phi(y) = \int_{y_0}^{y} \frac{1}{\sigma(x)} dx$, if $\sigma \in \mathcal{C}^1$ then by the Lamperti transform $Y_t = \phi(X_t)$ solves

$$dY_t = L(X_t)dt + dW_t, \quad Y_0 = \phi(x),$$

with $L(x) = \left(\frac{b}{\sigma} - \frac{\sigma'}{2}\right)(\phi^{-1}(x))$.

- As the function ϕ is monotonic, we get $\pi_{\mathcal{B}_D} = \pi_{\mathcal{D}}$ and $\pi_{\mathcal{B}_U} = \pi_{\mathcal{U}}$ where

$$\pi_{\mathcal{D}} = \mathbb{E}\left[g(Y_T)1_{\{\inf_{t\in[0,T]} Y_t > \mathcal{D}\}}\right], \ \pi_{\mathcal{U}} = \mathbb{E}\left[g(Y_T)1_{\{\sup_{t\in[0,T]} Y_t < \mathcal{U}\}}\right],$$

$g(x) = f \circ \phi^{-1}(x)$, $\mathcal{D} = \phi(\mathcal{B}_D)$ and $\mathcal{U} = \phi(\mathcal{B}_U)$.

- In the sequel, we consider the general setting given in [Alfonsi 2013] and let $(Y_t)_{t\geq 0}$ denote the SDE defined on $I = (0, +\infty)$ solution to

$$dY_t = L(Y_t)dt + \gamma dW_t, \quad t \geq 0, \; Y_0 = y \in I, \text{ with } \gamma \in \mathbb{R}^*, \qquad (1)$$

where $L : I \longrightarrow \mathbb{R}$ *is* C^2, s.t.

$$\exists\, \kappa > 0, \quad \forall y, y' \in I, y \leq y', L(y') - L(y) \leq \kappa(y' - y).$$

- In addition, for an arbitrary point $d \in I$, we assume that

$$v(x) = \int_d^x \int_d^y \exp\left(-\frac{2}{\gamma^2} \int_z^y L(\xi)d\xi \right) dzdy \;\text{ satisfies }\; \lim_{x \to 0^+} v(x) = +\infty. \qquad \text{(H1)}$$

- By the Feller's test, the above assumptions ensure the existence of a unique strong solution $(Y_t)_{t\geq 0}$ on $(0, +\infty)$.

The drift implicit Euler scheme

- For $t_i = \frac{iT}{n}$, $0 \leq i \leq n$, we consider the drift implicit continuous scheme introduced in [Alfonsi 2013] ,

$$\begin{aligned} \widehat{Y}^n_t &= \widehat{Y}^n_{t_i} + L(\widehat{Y}^n_t)(t - t_i) + \gamma(W_t - W_{t_i}), t \in [t_i, t_{i+1}] \\ \widehat{Y}^n_0 &= y \end{aligned} \tag{2}$$

is well defined and for all $t \in [0, T]$, $\hat{Y}^n_t \in I = (0, +\infty)$.

- If in addition we assume that for $p \geq 1$, we have

$$\mathbb{E}\Big[\Big(\int_0^T |L'(Y_u)L(Y_u) + \frac{\gamma^2}{2}L''(Y_u)|du\Big)^p\Big] < \infty \quad and \quad \mathbb{E}\Big[\Big(\int_0^T (L'(Y_u))^2 du\Big)^{\frac{p}{2}}\Big] < \infty, \tag{H2}$$

then by [Alfonsi 2013], there exists a positive constant K_p such that

$$\mathbb{E}^{\frac{1}{p}}\left[\sup_{t\in[0,T]} |\widehat{Y}^n_t - Y_t|^p\right] \leq K_p \frac{T}{n}.$$

The interpolated drift implicit scheme for Brownian bridge

- For our purpose, we rather focus on a slightly different interpolated version. For $t_i = \frac{iT}{n}$, $0 \leq i \leq n$,

$$\begin{cases} \overline{Y}^n_{t_{i+1}} = \overline{Y}^n_{t_i} + L(\overline{Y}^n_{t_{i+1}})\frac{T}{n} + \gamma(W_{t_{i+1}} - W_{t_i}), \\ \overline{Y}^n_0 = y. \end{cases} \quad (3)$$

and then introduce the following interpolated drift implicit scheme

$$\overline{Y}^n_t = \overline{Y}^n_{t_i} + L(\overline{Y}^n_{t_{i+1}})(t - t_i) + \gamma(W_t - W_{t_i}), \qquad \text{for } t \in [t_i, t_{i+1}[. \quad (4)$$

- The main advantages of this Brownian interpolation is that it preserves the rate of strong convergence of the original drift implicit scheme (2) and allows an easy use of of the Brownian bridge technique for pricing Barrier options.

Strong convergence rate

For this aim, we strengthen our assumption on L as follows:

$$L : I \to \mathbb{R} \text{ is } \mathcal{C}^2 \text{ such that: } L \text{ is decreasing on } (0, A) \text{ for } A > 0, \text{ and } L' \text{ satisfies } \exists L'_A > 0 \text{ s.t. } \forall y \in (A, \infty),\ |L'(y)| \leq L'_A. \quad \text{(H3)}$$

Theorem 1

Assume that conditions (H2) and (H3) hold true for a given $p > 1$, then

$$\mathbb{E}^{\frac{1}{p}}\Big[\sup_{t \in [0,T]} |\overline{Y}^n_t - Y_t|^p\Big] \leq K_p \frac{T}{n}.$$

Corollary 2

Under assumptions of Theorem 1, if in addition

$$\exists\, \alpha > 0 \text{ such that } \forall\, y \in I,\ \ yL(y) \leq \alpha(1 + |y|^2) \quad \text{(H4)}$$

then $\mathbb{E}[\sup_{0 \leq t \leq T} |\overline{Y}^n_t|^p] < \infty$.

Brownian bridge and drift implicit scheme

- The above barrier option prices can be approximated by

$$\overline{\pi}_{\mathcal{D}} := \mathbb{E}\Big[g(\overline{Y}^n_T)\prod_{i=0}^{n-1} 1_{\{\inf_{t\in[t_i,t_{i+1}]}\overline{Y}^n_t>\mathcal{D}\}}\Big] \text{ and } \overline{\pi}_{\mathcal{U}} := \mathbb{E}\Big[g(\overline{Y}^n_T)\prod_{i=0}^{n-1} 1_{\{\sup_{t\in[t_i,t_{i+1}]}\overline{Y}^n_t<\mathcal{U}\}}\Big].$$

- To get more accurate approximations, we use the Brownian bridge technique. For $x \in \mathbb{R}$, $(x)_+$ stands for $\max(x, 0)$.

Proposition 1

Under the above notation, for $h = \frac{T}{n}$, *we have*

$$\overline{\pi}_{\mathcal{D}} = \mathbb{E}\Big[g(\overline{Y}^n_T)\prod_{i=0}^{n-1}(1-\overline{q}_i)\Big], \overline{q}_i := \exp\left(\frac{-2(\overline{Y}^n_{t_i}-\mathcal{D})_+(\overline{Y}^n_{t_{i+1}}-\mathcal{D})_+}{\gamma^2 h}\right)$$

and

$$\overline{\pi}_{U} = \mathbb{E}\Big[g(\overline{Y}^n_T)\prod_{i=0}^{n-1}(1-\overline{p}_i)\Big], \overline{p}_i = \exp\left(\frac{-2(\mathcal{U}-\overline{Y}^n_{t_i})_+(\mathcal{U}-\overline{Y}^n_{t_{i+1}})_+}{\gamma^2 h}\right).$$

The Brownian bridge technic goes back to [Baldi 1995] and [Gobet 2009] for related refinements.

The interpolated drift implicit Euler MLMC method

- We consider the drift implicit scheme $(\overline{Y}^{2^\ell}_{t_i})_{0\leq i\leq 2^\ell}$ given in (3) using a time step $h_\ell = 2^{-\ell}T$ for $\ell \in \{0, ..., L\}$, with $L = \log n/\log 2$, where n is the finest time step number.

- Let $(\overline{Y}^{2^\ell}_t)_{0\leq t\leq T}$ denote the Brownian interpolation of the drift implicit scheme defined in (4) with time step h_ℓ. For

$$\overline{P}_\ell := g(\overline{Y}^{2^\ell}_T)\prod_{i=0}^{2^\ell-1} 1_{\{\sup_{t\in[t^\ell_i,t^\ell_{i+1}]}\overline{Y}^{2^\ell}_t<\mathcal{U}\}}, \quad \text{where } t^\ell_i = \frac{iT}{2^\ell} \quad \text{for } \ell\in\{0,...,L\}, \tag{5}$$

we have

$$\overline{\pi}_\mathcal{U} = \mathbb{E}\big[\overline{P}_L\big] = \mathbb{E}\big[\overline{P}_0\big] + \sum_{\ell=1}^{L}\mathbb{E}\big[\overline{P}_\ell - \overline{P}_{\ell-1}\big], \tag{6}$$

where $\overline{\pi}_\mathcal{U} := \mathbb{E}\Big[g(\overline{Y}^n_T)\prod_{i=0}^{n-1} 1_{\{\sup_{t\in[t_i,t_{i+1}]}\overline{Y}^n_t<\mathcal{U}\}}\Big]$.

- Applying Proposition 2 yields

$$\mathbb{E}\left[\overline{P}_\ell\right] = \mathbb{E}\left[\overline{P}^f_\ell\right], \text{ where } \overline{P}^f_\ell := g(\overline{Y}^{2^\ell}_T) \prod_{i=0}^{2^\ell-1} (1 - \overline{p}^{2^\ell}_i) \text{ with}$$

$$\overline{p}^{2^\ell}_i = \exp\left(\frac{-2(\mathcal{U} - \overline{Y}^{2^\ell}_{t^\ell_i})_+(\mathcal{U} - \overline{Y}^{2^\ell}_{t^\ell_{i+1}})_+}{\gamma^2 h_\ell}\right).$$

- Following the conditional MC proposed by [Giles et al. 2019]

$$\mathbb{E}[\overline{P}_{\ell-1}] = \mathbb{E}[g(\overline{Y}^{2^{\ell-1}}_T) \prod_{i=0}^{2^{\ell-1}-1} \mathbb{E}[1_{\{\sup_{t\in[t^{\ell-1}_i, t^{\ell-1}_{i+1}]} \overline{Y}^{2^{\ell-1}}_t < \mathcal{U}\}} | \overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_i}, \overline{Y}^{2^{\ell-1}}_{t^\ell_{2i+1}}, \overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_{i+1}}]] =$$

$$\mathbb{E}[g(\overline{Y}^{2^{\ell-1}}_T) \prod_{i=0}^{2^{\ell-1}-1} \mathbb{E}[1_{\{\sup_{t\in[t^{\ell-1}_i, t^\ell_{2i+1}]} \overline{Y}^{2^{\ell-1}}_t < \mathcal{U}\}} 1_{\{\sup_{t\in[t^\ell_{2i+1}, t^{\ell-1}_{i+1}]} \overline{Y}^{2^{\ell-1}}_t < \mathcal{U}\}} | \overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_i}, \overline{Y}^{2^{\ell-1}}_{t^\ell_{2i+1}}, \overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_{i+1}}]],$$

where the coarse scheme $\overline{Y}^{2^{\ell-1}}_{t^\ell_{2i+1}}$ is computed using our Brownian interpolation scheme (4) that is

$$\overline{Y}^{2^{\ell-1}}_{t^\ell_{2i+1}} = \overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_i} + L(\overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_{i+1}})(t^\ell_{2i+1} - t^{\ell-1}_i) + \gamma(W_{t^\ell_{2i+1}} - W_{t^{\ell-1}_i}).$$

Brownian bridge for the MLMC method (continued)

- Thus, we rewrite $\sup_{t\in[t_i^{\ell-1},t_{2i+1}^{\ell}]} \overline{Y}_t^{2^{\ell-1}}$ and $\sup_{t\in[t_{2i+1}^{\ell},t_{i+1}^{\ell-1}]} \overline{Y}_t^{2^{\ell-1}}$ as follows

$$\sup_{t\in[t_i^{\ell-1},t_{2i+1}^{\ell}]} \overline{Y}_t^{2^{\ell-1}} = \overline{Y}_{t_i^{\ell-1}}^{2^{\ell-1}} + \gamma \sup_{t\in[t_i^{\ell-1},t_{2i+1}^{\ell}]} \left(W_t - W_{t_i^{\ell-1}} + \frac{1}{\gamma} L(\overline{Y}_{t_{i+1}^{\ell-1}}^{2^{\ell-1}})(t - t_i^{\ell-1}) \right)$$

with

$$W_{t_{2i+1}^{\ell}} - W_{t_i^{\ell-1}} + \tfrac{1}{\gamma} L(\overline{Y}_{t_{i+1}^{\ell}}^{2^{\ell-1}})(t_{2i+1}^{\ell} - t_i^{\ell-1}) = \tfrac{1}{\gamma} \left(\overline{Y}_{t_{2i+1}^{\ell}}^{2^{\ell-1}} - \overline{Y}_{t_i^{\ell-1}}^{2^{\ell-1}} \right)$$

and

$$\sup_{t\in[t_{2i+1}^{\ell},t_{i+1}^{\ell-1}]} \overline{Y}_t^{2^{\ell-1}} = \overline{Y}_{t_{2i+1}^{\ell}}^{2^{\ell-1}} + \gamma \sup_{t\in[t_i^{\ell-1},t_{2i+1}^{\ell}]} \left(W_t - W_{t_{2i+1}^{\ell}} + \frac{1}{\gamma} L(\overline{Y}_{t_{i+1}^{\ell-1}}^{2^{\ell-1}})(t - t_{2i+1}^{\ell}) \right)$$

with

$$W_{t_{i+1}^{\ell-1}} - W_{t_{2i+1}^{\ell}} + \tfrac{1}{\gamma} L(\overline{Y}_{t_{i+1}^{\ell-1}}^{2^{\ell-1}})(t_{i+1}^{\ell-1} - t_{2i+1}^{\ell}) = \tfrac{1}{\gamma} \left(\overline{Y}_{t_{i+1}^{\ell-1}}^{2^{\ell-1}} - \overline{Y}_{t_{2i+1}^{\ell}}^{2^{\ell-1}} \right).$$

Brownian bridge for the MLMC method (continued)

Then, using the Girsanov theorem, we get

$$\mathbb{E}\left[\overline{P}_{\ell-1}\right] = \mathbb{E}\left[\overline{P}^c_{\ell-1}\right], \text{ where } \overline{P}^c_{\ell-1} := g(\overline{Y}^{2^{\ell-1}}_T) \prod_{i=0}^{2^{\ell-1}-1} (1 - \overline{p}^{2^{\ell-1}}_{i,1})(1 - \overline{p}^{2^{\ell-1}}_{i,2})$$

with

$$\overline{p}^{2^{\ell-1}}_{i,1} = \exp\left(\frac{-2(\mathcal{U} - \overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_i})_+(\mathcal{U} - \overline{Y}^{2^{\ell-1}}_{t^{\ell}_{2i+1}})_+}{\gamma^2 h_\ell}\right),$$

$$\overline{p}^{2^{\ell-1}}_{i,2} = \exp\left(\frac{-2(\mathcal{U} - \overline{Y}^{2^{\ell-1}}_{t^{\ell}_{2i+1}})_+(\mathcal{U} - \overline{Y}^{2^{\ell-1}}_{t^{\ell-1}_{i+1}})_+}{\gamma^2 h_\ell}\right),$$

which can be rewritten as

$$\overline{P}^c_{\ell-1} := g(\overline{Y}^{2^{\ell-1}}_T) \prod_{i=0}^{2^\ell-1} (1 - \overline{p}^{2^{\ell-1}}_i) \text{ with } \overline{p}^{2^{\ell-1}}_i = \exp\left(\frac{-2(\mathcal{U} - \overline{Y}^{2^{\ell-1}}_{t^\ell_i})_+(\mathcal{U} - \overline{Y}^{2^{\ell-1}}_{t^\ell_{i+1}})_+}{\gamma^2 h_\ell}\right), \tag{7}$$

where the coarse scheme $\overline{Y}^{2^{\ell-1}}_{t^\ell_i}$ evaluated over the finest time grid is computed using the Brownian interpolation scheme (4).

- Thus, the improved MLMC method approximates $\overline{\pi}_{\mathcal{U}}$ by

$$\bar{P}_{\mathcal{U}} := \frac{1}{N_0}\sum_{k=1}^{N_0}\overline{P}^f_{0,k} + \sum_{\ell=1}^{L}\frac{1}{N_\ell}\sum_{k=1}^{N_\ell}\left(\overline{P}^f_{\ell,k} - \overline{P}^c_{\ell-1,k}\right), \tag{8}$$

where the condition $\mathbb{E}\left[\overline{P}^f_{\ell-1}\right] = \mathbb{E}\left[\overline{P}^c_{\ell-1}\right]$ is satisfied.

- Similarly, the improved MLMC method approximates $\overline{\pi}_{\mathcal{D}}$ by

$$\bar{Q}_{\mathcal{D}} := \frac{1}{N_0}\sum_{k=1}^{N_0}\overline{Q}^f_{0,k} + \sum_{\ell=1}^{L}\frac{1}{N_\ell}\sum_{k=1}^{N_\ell}\left(\overline{Q}^f_{\ell,k} - \overline{Q}^c_{\ell-1,k}\right), \tag{9}$$

where

$$\overline{Q}^f_{\ell} := g(\overline{Y}^{2^\ell}_T)\prod_{i=0}^{2^\ell-1}(1-\overline{q}^{2^\ell}_i) \text{ with } \overline{q}^{2^\ell}_i = \exp\left(\frac{-2(\overline{Y}^{2^\ell}_{t^\ell_i}-\mathcal{D})_+(\overline{Y}^{2^\ell}_{t^\ell_{i+1}}-\mathcal{D})_+}{\gamma^2 h_\ell}\right)$$

$$\overline{Q}^c_{\ell-1} := g(\overline{Y}^{2^{\ell-1}}_T)\prod_{i=0}^{2^\ell-1}(1-\overline{q}^{2^{\ell-1}}_i) \text{ with } \overline{q}^{2^{\ell-1}}_i = \exp\left(\frac{-2(\overline{Y}^{2^{\ell-1}}_{t^\ell_i}-\mathcal{D})_+(\overline{Y}^{2^{\ell-1}}_{t^\ell_{i+1}}-\mathcal{D})_+}{\gamma^2 h_\ell}\right)$$

Complexity Theorem [Giles, 2008]

Let P denote a payoff functional and $\hat{P}_l$ denote the corresponding approximation using a numerical discretisation with time step $h_\ell = \frac{T}{M^\ell}$.

$$\hat{Y} = \sum_{\ell=0}^{L} \hat{Y}_\ell = \frac{1}{N_0} \sum_{i=1}^{N_0} \hat{P}_0^{(i)} + \sum_{\ell=1}^{L} \frac{1}{N_\ell} \sum_{i=1}^{N_\ell} \left(\hat{P}_\ell^{(i)} - \hat{P}_{\ell-1}^{(i)} \right)$$

Assume that

1. $\mathbb{E}[\hat{P}_\ell - P] \leq c_1 h_\ell^\alpha$
2. $\mathbb{E}[\hat{Y}_\ell] = \begin{cases} \mathbb{E}[\hat{P}_0], \\ \mathbb{E}[\hat{P}_\ell - \hat{P}_{\ell-1}] \end{cases}$
3. $\mathrm{Var}[\hat{Y}_\ell] \leq c_2 N_\ell^{-1} h_\ell^\beta$
4. C_l, the computational complexity of $\hat{Y}_\ell$, is bounded by
$$C_\ell \leq c_3 N_\ell h_\ell^{-1},$$

Complexity Theorem

- Then, for any $\varepsilon > 0$ small enough there are values L et N_ℓ for which

$$MSE \equiv \mathbb{E}\left[(\hat{Y} - \mathbb{E}[P])^2\right] < \varepsilon^2,$$

with a computational complexity C_{MLMC} with bound

$$C_{\mathrm{MLMC}} = \begin{cases} O(\varepsilon^{-2}), & \beta > 1 \\ O(\varepsilon^{-2}(\log \epsilon)^2), & \beta = 1 \\ O(\varepsilon^{-2-(1-\beta)/\alpha}), & 0 < \beta < 1. \end{cases}$$

- Let us recall that for the same MSE of order ε^2 the optimal complexity of the classical Monte Carlo is

$$C_{\mathrm{MC}} = O(\varepsilon^{-3})$$

Extreme path events

For $p \geq 1$, assumption (H2) is valid and $\sup_{t \in [0,T]} \mathbb{E}\left[|L(Y_t)|^p \right] < \infty.$ $(\tilde{\mathrm{H}}2)$

Lemma 3

Assume that conditions $(\tilde{\mathrm{H}}2)$, (H3) *and* (H4) *are satisfied for a given* $p > 1$ *and* $0 < L'_A < \frac{1}{2h_\ell}$, *with* $h_\ell = 2^{-\ell} T$ *sufficiently small. Let* $\eta \in (0,1)$, *then*

$$\mathbb{P}\left(\max\left(\sup_{0 \leq i \leq 2^\ell} (|Y_{t_i^\ell}|, |\overline{Y}^{2^\ell}_{t_i^\ell}|, |\overline{Y}^{2^{\ell-1}}_{t_i^\ell}|) \right) > h_\ell^{-\eta} \right) = o(h_\ell^q)$$

$$\mathbb{P}\left(\max\left(\sup_{0 \leq i \leq 2^\ell} \left(|Y_{t_i^\ell} - \overline{Y}^{2^\ell}_{t_i^\ell}|, |Y_{t_i^\ell} - \overline{Y}^{2^{\ell-1}}_{t_i^\ell}|, |\overline{Y}^{2^\ell}_{t_i^\ell} - \overline{Y}^{2^{\ell-1}}_{t_i^\ell}| \right) \right) > h_\ell^{1-\eta} \right) = o(h_\ell^q)$$

$$\sup_{0 \leq i \leq 2^\ell} \mathbb{P}\left(\int_{t_i^\ell}^{t_{i+1}^\ell} |L(Y_s)| ds > h_\ell^{1-\eta} \right) = o(h_\ell^q) \quad \textit{for all } 0 < q < p\eta, \textit{ and}$$

$$\sup_{0 \leq i \leq 2^\ell} \mathbb{P}\left(\sup_{t \in [t_i^\ell, t_{i+1}^\ell]} |W_t - W_{t_i^\ell}| > h_\ell^{\frac{1}{2} - \eta} \right) = o(h_\ell^q), \quad \textit{for all } q > 0.$$

Theorem 4

Let g denote a payoff function satisfying : $\exists C > 0$ s.t. $\forall x, y > 0$,

$$|g(x) - g(y)| \leq C|x - y|(1 + |x|^\nu + |y|^\nu) \text{ and } |g(x)| \leq C(1 + |x|^{\nu+1}),$$

with $\nu > 0$.

Moreover, assume that conditions $(\tilde{\mathrm{H}}2)$, (H3) and (H4) are satisfied for

$$p > \frac{(1+\delta)(1+\gamma)[7(1+\varepsilon)+2\nu]}{\frac{1}{2}-\delta} \text{ with } \varepsilon, \gamma > 0,\ \delta \in (0, 1/2).$$

If in addition $\inf_{t\in[0,T]} Y_t$ (resp. $\sup_{t\in[0,T]} Y_t$) has a bounded density in the neighborhood of the barrier $\mathcal{D}$ (resp. $\mathcal{U}$), then

$$\mathrm{Var}(\overline{\mathrm{Q}}^{\mathrm{f}}_\ell - \overline{\mathrm{Q}}^{\mathrm{c}}_\ell) = \mathrm{O}(\mathrm{h}_\ell^{1+\delta}) \quad (\textit{resp.} \mathrm{Var}(\overline{\mathrm{P}}^{\mathrm{f}}_\ell - \overline{\mathrm{P}}^{\mathrm{c}}_\ell) = \mathrm{O}(\mathrm{h}_\ell^{1+\delta})).$$

Time complexity analysis

- Combining the complexity theorem in [Giles 2008] with the above result, we deduce that for any $\delta \in (0, \frac{1}{2})$ the MLMC estimators $\bar{Q}_{\mathcal{D}}$ and $\bar{P}_{\mathcal{U}}$ reach the optimal time complexity $O(\varepsilon^{-2})$, for a given precision $\varepsilon > 0$, and behave like an unbiased Monte Carlo estimator.

- Taking δ close to $\frac{1}{2}$ achieves a smaller variance of the difference between the finer and coarse approximations which is of order $O(h_\ell^{\beta})$ with β close to $\frac{3}{2}$ similar to the case of diffusion with Lipschitz coefficients studied in [Giles et al. 2019], but clearly leads to very restrictive conditions on the finiteness of the moments of $(Y_t)_{t\in[0,T]}$ and $(\bar{Y}_t^n)_{t\in[0,T]}$.

Sketch of the proof

- **First event** A_1We consider any of the extreme path events. For $\gamma > 0$

$$\mathbb{E}[(\overline{Q}^f_\ell - \overline{Q}^c_\ell)^2 1_{A_1}] \leq 2^{\frac{2+\gamma}{1+\gamma}} \left(\mathbb{E}^{\frac{\gamma}{1+\gamma}}[|\overline{Q}^f_\ell|^{\frac{2(1+\gamma)}{\gamma}}] + \mathbb{E}^{\frac{\gamma}{1+\gamma}}[|\overline{Q}^c_\ell|^{\frac{2(1+\gamma)}{\gamma}}]\right) \left(\mathbb{P}[A_1]\right)^{\frac{1}{1+\gamma}}.$$

By Lemma 3, we get that

$$\mathbb{E}[(\overline{Q}^f_\ell - \overline{Q}^c_\ell)^2 1_{A_1}] = o(h_\ell^{\frac{q}{1+\gamma}}) \text{ for all } q \text{ such that } 0 < \frac{q}{\eta} \leq p.$$

- **Second event** A_2 corresponds to the non-extreme paths satisfying

$$|\inf_{t\in[0,T]} Y_t - \mathcal{D}| > h_\ell{}^{\frac{1}{2}-\eta(1+\varepsilon)} \text{ for } \eta \in (0, 1/2(1+\varepsilon)) \text{ with } \varepsilon > 0.$$

➞ We prove that for h_ℓ sufficiently small $\prod_{i=0}^{2^\ell-1}(1-\overline{q}_i^{2^\ell})$ and $\prod_{i=0}^{2^\ell-1}(1-\overline{q}_i^{2^{\ell-1}})$ are both equal to $1 + o(h_\ell{}^a)$ for all $a > 0$.
Consequently, we deduce that

$$\mathbb{E}[(\overline{Q}^f_\ell - \overline{Q}^c_\ell)^2 1_{A_2}] = O(h_\ell{}^{2(1-\eta)-2\eta\nu}).$$

- **Third event** A_3 corresponds to the rest of the non extreme paths.

➡ We prove

$$\left|\prod_{i=0}^{2^\ell-1}(1-\overline{q}_i^{2^\ell})-\prod_{i=0}^{2^\ell-1}(1-\overline{q}_i^{2^{\ell-1}})\right|=O(h_\ell{}^{\frac{1}{2}-2\eta(1+\varepsilon)}).$$

Therefore, as we work on the non-extreme paths events, we deduce that

$$\mathbb{E}[(\overline{Q}_\ell^f-\overline{Q}_\ell^c)^2 1_{A_3}]=O(h_\ell{}^{1-6\eta(1+\varepsilon)-2\eta\nu}\times\mathbb{P}(|\inf_{t\in[0,T]}Y_t-\mathcal{D}|\leq h_\ell{}^{\frac{1}{2}-\eta(1+\varepsilon)}))$$
$$=O(h_\ell{}^{\frac{3}{2}-7\eta(1+\varepsilon)-2\eta\nu})$$

as the random variable $\inf_{t\in[0,T]}Y_t$ has a bounded density on the neighborhood of D. To complete the proof, we choose

$$\eta=\frac{\frac{1}{2}-\delta}{7(1+\varepsilon)+2\nu},$$

which yields $\mathbb{E}[(\overline{Q}_\ell^f-\overline{Q}_\ell^c)^2 1_{A_3}]=O(h_\ell^{1+\delta})$.

- Putting all this together gives the result as soon as

$$p>\frac{(1+\delta)(1+\gamma)[7(1+\varepsilon)+2\nu]}{\frac{1}{2}-\delta}.$$

Application to the CIR model

- we consider the problem of pricing D-O and U-O barrier options

$$\pi_{\mathcal{D}} = \mathbb{E}\left[f(X_T)1_{\{\inf_{t\in[0,T]} X_t > \mathcal{D}\}}\right] \text{ and } \pi_{\mathcal{U}} = \mathbb{E}\left[f(X_T)1_{\{\sup_{t\in[0,T]} X_t < \mathcal{U}\}}\right],$$

where f is a Lipschitz payoff function with Lipschitz and

$$dX_t = (a - \kappa X_t)dt + \sigma\sqrt{X_t}dW_t, \quad X_0 = x > 0 \qquad (10)$$

with $a \geq \sigma^2/2$, $\kappa \in \mathbb{R}$, $\sigma > 0$.

- Applying Lamperti transformation $Y_t = \sqrt{X_t}$ satisfies

$$dY_t = L(Y_t)dt + \gamma dW_t, \quad Y_0 = \sqrt{x},$$

where $L(y) = \dfrac{a - \sigma^2/4}{2y} - \dfrac{\kappa}{2}y$ and $\gamma = \dfrac{\sigma}{2}$.

- Thus, for $g : x \in \mathbb{R} \mapsto g(x) = f(x^2)$ we get

$$\pi_{\mathcal{D}} = \mathbb{E}\left[g(Y_T)1_{\{\inf_{t\in[0,T]} Y_t > \sqrt{\mathcal{D}}\}}\right] \text{ and } \pi_{\mathcal{U}} = \mathbb{E}\left[g(Y_T)1_{\{\sup_{t\in[0,T]} Y_t < \sqrt{\mathcal{U}}\}}\right].$$

- As $a - \sigma^2/4 > 0$, we easily check assumptions (H1), (H3) and (H4).
- To check $(\tilde{\mathrm{H}}2)$ it is enough to show that

$$\sup_{t\in[0,T]} \mathbb{E}\big[|L'(Y_t)L(Y_t)|^p + |L''(Y_t)|^p + |L'(Y_t)|^{(2\vee p)} + |L(Y_t)|^p\big] < \infty$$

which is clearly satisfied as soon as

$$\sup_{t\in[0,T]} \mathbb{E}\big[Y_t^{-(4\vee 3p)}\big] = \sup_{t\in[0,T]} \mathbb{E}\big[X_t^{-(2\vee \frac{3}{2}p)}\big] < \infty.$$

Recalling that $\sup_{t\in[0,T]} \mathbb{E}\big[X_t^q\big] < \infty$ for all $q > -\frac{2a}{\sigma^2}$ we easily conclude that this holds when $\sigma^2 < a$ and $p < \frac{4}{3}\frac{a}{\sigma^2}$.

- Consequently, for $\delta \in (0, 1/2)$, if

$$\frac{4}{3}\frac{a}{\sigma^2} > p > \frac{(1+\delta)(1+\gamma)[7(1+\varepsilon)+2\nu]}{\frac{1}{2}-\delta} > 18$$

- Then Theorem 4 is valid provided that $\inf_{t\in[0,T]} Y_t$ (resp. $\sup_{t\in[0,T]} Y_t$) has a bounded density in the neighborhood of the barrier.

Running maximum of the CIR process

- Let us introduce firstly the confluent hypergeometric function $_1F_1(x,b,y)$ defined for all $y, x \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0,-1,-2,\cdots\}$ by

$$_1F_1(x,b,y) = \sum_{n=0}^{\infty} \frac{(x)_n}{(b)_n n!} y^n,$$

where $(x)_n = x(x+1)...(x+n-1)$ stands for the Pochhammer symbol.

Theorem 5

Let $(X_t)_{0\le t\le T}$ denote the CIR process solution to (10). Then $\sup_{t\in[0,T]} X_t$ has a continuous density on any compact set $K \subset (X_0,+\infty)$, given by

$$z \in K \mapsto P_{\mathrm{CIR,Max}}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(1+iu)T}\hat{\phi}(u,z)du$$

with

$$\hat{\phi}(u,z) = \frac{_1F_1((1+iu)/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2)\,_1F_1((1+iu)/\kappa+1, 2a/\sigma^2+1, 2\kappa z/\sigma^2)}{a\,_1F_1((1+iu)/\kappa, 2a/\sigma^2, 2\kappa z/\sigma^2)^2}.$$

- To do so, we introduce the Tricomi confluent hypergeometric function $U(a,b,z)$ defined for all $a, z \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{\pm 0, \pm 1, \pm 2, \dots\}$ by

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} {}_1F_1(a,b,z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} {}_1F_1(1+a-b, 2-b, z).$$

- Let $\tau_{X_0 \downarrow z} := \inf\{t \geq 0 : X_t = z\}$ for $0 < z < X_0$.

$$\mathrm{E}[e^{-s\tau_{X_0\downarrow z}}] = \frac{U(s/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2)}{U(s/\kappa, 2a/\sigma^2, 2\kappa z/\sigma^2)}, \text{ for } s > 0.$$

Theorem 6

The running minimum $\inf_{t\in[0,T]} X_t$ has a continuous density on any compact set $K \subset (0, X_0)$, given by

$$z \in K \mapsto P_{\mathrm{CIR,Min}}(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(1+iu)T} \hat{\psi}(u,z) du$$

with

$$\hat{\psi}(u,z) = \frac{2U((1+iu)/\kappa, 2a/\sigma^2, 2\kappa X_0/\sigma^2) U((1+iu)/\kappa + 1, 2a/\sigma^2 + 1, 2\kappa z/\sigma^2)}{\sigma^2 U((1+iu)/\kappa, 2a/\sigma^2, 2\kappa z/\sigma^2)^2}.$$

Numerical Tests

- We consider the problem of pricing D-O and U-O barrier options $\pi_{\mathcal{D}} = \mathbb{E}\Big[f(X_T)1_{\{\inf_{t\in[0,T]} X_t > \mathcal{D}\}}\Big]$ and $\pi_{\mathcal{U}} = \mathbb{E}\Big[f(X_T)1_{\{\sup_{t\in[0,T]} X_t < \mathcal{U}\}}\Big]$, where the payoff function $f(x) = e^{-rT}(x-K)_+$.

- By the Lamperti transform we get

$\pi_{\mathcal{D}} = \mathbb{E}\Big[g(Y_T)1_{\{\inf_{t\in[0,T]} Y_t > \sqrt{\mathcal{D}}\}}\Big]$ and $\pi_{\mathcal{U}} = \mathbb{E}\Big[g(Y_T)1_{\{\sup_{t\in[0,T]} Y_t < \sqrt{\mathcal{U}}\}}\Big]$, where $g(x) = e^{-rT}(x^2-K)_+$ and $(Y_t)_{t\in[0,T]}$.

- We consider our interpolated drift implicit scheme

$$\overline{Y}^n_t = \overline{Y}^n_{t_i} + \left(\frac{a-\gamma^2}{2\overline{Y}^n_{t_{i+1}}} - \frac{\kappa}{2}\overline{Y}^n_{t_{i+1}}\right)(t-t_i) + \gamma(W_t - W_{t_i}), \qquad \text{for } t \in [t_i, t_{i+1}],$$

$Y_0 = \sqrt{X_0}$ and $\gamma = \dfrac{\sigma}{2}$.

For n large enough, the positive solution is

$$\overline{Y}^n_{t_{i+1}} = \frac{\sqrt{(2+\kappa\frac{T}{n})(a-\gamma^2)\frac{T}{n} + (\gamma(W_{t_{i+1}} - W_{t_i}) + \overline{Y}^n_{t_i})^2} + \gamma(W_{t_{i+1}} - W_{t_i}) + \overline{Y}^n_{t_i}}{2+\kappa\frac{T}{n}}.$$

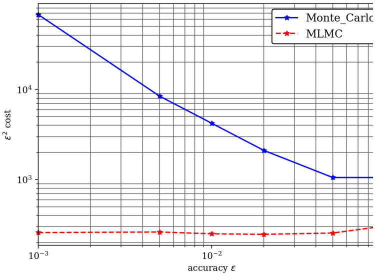
- We take $r = 0.1$, $X_0 = 100$, $a = 0$, $\kappa = -0.1$, $\sigma = 2.5$ and $T = 0.5$. For the D-O option the strike is $K = 95$, and the barrier $\mathcal{D} = 90$ and for the U-O option the strike is $K = 105$ and the barrier $\mathcal{U} = 120$.
- The benchmark prices given in [Davydov and Linetsky 2001] for the D-O (resp. U-O) option is 10.6013 (resp. 0.7734).
- The performance of the improved MLMC is given in the tables and figure below.

Accuracy	Price	MLMC cost	MC cost	Saving
10^{-3}	10.669	2.588×10^8	6.752×10^{10}	260.91
5×10^{-3}	10.668	1.051×10^7	3.376×10^8	32.13
10^{-2}	10.668	2.510×10^6	4.220×10^7	16.81
2×10^{-2}	10.677	6.187×10^5	5.275×10^6	8.52

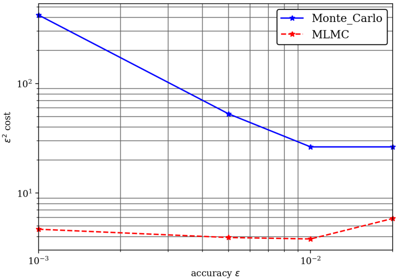
Table: MLMC complexity tests for D-O barrier option pricing of $\pi_{\mathcal{D}}$

Accuracy	Price	MLMC cost	MC cost	Saving
10^{-3}	0.77200	4.674×10^{6}	4.221×10^{8}	90.32
5×10^{-3}	0.76926	1.571×10^{5}	2.11×10^{6}	13.44
10^{-2}	0.77015	3.809×10^{4}	2.638×10^{5}	6.93
2×10^{-2}	0.78168	1.463×10^{4}	6.596×10^{4}	4.51

Table: MLMC complexity tests for U-O barrier option pricing $\pi_{\mathcal{U}}$



(a) Approximation of $\pi_{\mathcal{D}}$



(b) Approximation of $\pi_{\mathcal{U}}$

Figure: Comparison for the performances of MLMC vs classical MC algorithm

Application to the CEV model

- For CEV process solution to

$$dX_t = \mu X_t dt + \sigma X_t^{\alpha} dW_t, \quad t \geq 0, \quad , X_0 > 0, \quad \mu \in \mathbb{R} \quad \text{and } \alpha > 1$$

we consider the problem of pricing an U-O barrier option

$$\Pi_{\mathcal{D}}^{\text{U-O},X} = \mathbb{E}\Big[f(X_T)1_{\{\sup_{t\in[0,T]} X_t < \mathcal{D}\}}\Big], \Pi_{\mathcal{U}}^{\text{D-O},X} := \mathbb{E}\Big[f(X_T)1_{\{\inf_{t\in[0,T]} X_t > \mathcal{U}\}}\Big].$$

- For $\alpha > 1$, by Feller's test the solution $(X_t)_{t\in[0,T]}$ is positive.

- So applying the Lamperti transformation, $Y_t = X_t^{1-\alpha}$ is well defined on $I = (0, +\infty)$ and satisfies

$$dY_t = L(Y_t)dt + \gamma dW_t, \text{ where } L(y) = (1-\alpha)\left(\mu y - \alpha \frac{\sigma^2}{2} y^{-1}\right)$$

and thus

$$\Pi_{\mathcal{D}}^{\text{U-O},X} = \mathbb{E}\Big[g(Y_T)1_{\{\inf_{t\in[0,T]} Y_t > \mathcal{D}^{1-\alpha}\}}\Big] \Pi_{\mathcal{U}}^{\text{D-O},X} = \mathbb{E}\Big[g(Y_T)1_{\{\sup_{t\in[0,T]} Y_t < \mathcal{U}^{1-\alpha}\}}\Big]$$

with $g : x \in \mathbb{R} \mapsto f(x^{\frac{1}{1-\alpha}})$.

Application to the CEV model

- we easily check assumptions (H1), (H3) and (H4)

- On the one hand, by Itô's formula the process $(Z_t)_{0 \leq t \leq T}$ given by $Z_t = \frac{X_t^{-2(\alpha-1)}}{4(\alpha-1)^2}$ is a CIR process solution to

$$dZ_t = (a - \kappa Z_t)dt - \sigma\sqrt{Z_t}dW_t, \text{ with } a = \frac{\sigma^2(2\alpha-1)}{4(\alpha-1)} \text{ and } \kappa = 2\mu(\alpha-1).$$

- We deduce that $\sup_{t\in[0,T]} \mathbb{E}[Y_t^q] < \infty$ for $q > -\frac{2\alpha-1}{2(\alpha-1)}$.

- On the other hand to check assumption $(\tilde{\mathrm{H}}2)$ it is enough to show that

$$\sup_{t\in[0,T]} \mathbb{E}\left[|L'(Y_t)L(Y_t)|^p + |L''(Y_t)|^p + |L'(Y_t)|^{(2\vee p)} + |L(Y_t)|^p\right] < \infty$$

which is satisfied if $\sup_{t\in[0,T]} \mathbb{E}\left[Y_t^{-(4\vee 3p)}\right] < \infty$.

- This condition is satisfied when $4 < \frac{2\alpha-1}{2(\alpha-1)}$ (i.e. $\alpha \in (1, \frac{7}{6})$) and $p < \frac{2\alpha-1}{6(\alpha-1)}$.

Application to the CEV model

- Besides, we have

$$|g(x)-g(y)| \leq \frac{[f]_{\text{Lip}}}{\alpha-1}|x-y|(|x|^{-\frac{\alpha}{\alpha-1}}+|y|^{-\frac{\alpha}{\alpha-1}}), \text{ for all } x,y \in I=(0,+\infty).$$

Theorem 7

Let g denotes a bounded payoff function satisfying :
$\exists C>0$ *s.t.* $\forall x,y>0$,

$$|g(x)-g(y)| \leq C|x-y|(1+|x|^{-\nu}+|x|^{-\nu}), \text{ for } \nu>0.$$

Moreover, assume that conditions $(\tilde{\mathrm{H}}2)$, (H3) *and* (H4) *are satisfied for*

$$p > \frac{(1+\delta)(1+\gamma)5(1+\varepsilon)}{\frac{1}{2}-\delta}, \text{ with } \varepsilon,\gamma>0 \text{ and } \delta \in (0,1/2).$$

If in addition $\inf_{t\in[0,T]} Y_t$ *has a bounded density in the neighborhood of the barrier, then*

$$\mathrm{Var}(\overline{\mathrm{Q}}^{\mathrm{f}}_{\ell}-\overline{\mathrm{Q}}^{\mathrm{c}}_{\ell}) = \mathrm{O}(\mathrm{h}_{\ell}^{1+\delta}).$$

Finally, for $\delta \in (0, 1/2)$, if we choose α such that $1 < \alpha < \frac{59}{58} < \frac{7}{6}$ then we can find p such that $\frac{2\alpha-1}{6(\alpha-1)} > p > \frac{(1+\delta)(1+\gamma)5(1+\varepsilon)}{\frac{1}{2}-\delta} > 10$.

Thus Theorem 7 is valid provided that $\inf_{t\in[0,T]} Y_t$ has a bounded density in the neighborhood of the barrier $\mathcal{D}^{1-\alpha}$.

Remark.

One can also consider the CEV process for $\alpha \in (\frac{1}{2}, 1)$ solution to

$$dX_t = (a - \kappa X_t)dt + \sigma Y_t^{\alpha} dW_t, \text{ with } X_0 > 0, a > 0.$$

It can be easily checked that for $a > 0$ this SDE is well defined on $I = (0, +\infty)$. However, the condition that $\inf_{t\in[0,T]} X_t$ or $\sup_{t\in[0,T]} X_t$ admits a continuous density in the neighborhood of the barrier seems to be a challenging problem.

- Let us denote by $\tau_{X_0\uparrow z} := \inf\{t \geq 0 : X_t = z\}$ the first time that the CEV process $(X_t)_{t\geq 0}$ starting at X_0 hits the level $z > X_0$.

- From [Jeanblanc, Yor and Chesney 2009], the Laplace transform of the hitting time $\tau_{X_0\uparrow z}$ is given by

$$\mathrm{E}[e^{-s\tau_{X_0\uparrow z}}] = \left(\frac{X_0}{z}\right)^{\beta+\frac{1}{2}} \exp\left(\frac{\epsilon}{2}c(X_0^{-2\beta} - z^{-2\beta})\right)\frac{W_{k,n}(cX_0^{-2\beta})}{W_{k,n}(cz^{-2\beta})},$$

with $\epsilon = \mathrm{sign}(\mu\beta)$, $n = \frac{1}{4\beta}$, $k = \epsilon\left(\frac{1}{2} + \frac{1}{4\beta}\right) - \frac{s}{2|\mu\beta|}$ and $W_{k,n}$ the Whittaker's function

$$W_{k,n}(y) = y^{n+\frac{1}{2}}e^{-y/2}U(n-k+\frac{1}{2}, 2n+1, y),$$

where U denotes the confluent hypergeometric function of second kind and $\beta = \alpha - 1$ and $c = \dfrac{|\mu|}{\beta\sigma^2}$.

Theorem 8

Let $(X_t)_{0\leq t\leq T}$ denotes the CEV process. Then $\sup_{t\in[0,T]} X_t$ *has a continuous density on any compact set* $K \subset (X_0, +\infty)$*, given by*

$$z \in K \mapsto P_{CEV,\ Max}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(1+iu)T}\hat{\Phi}(z,u)du,$$

with

$$\hat{\Phi}(z,u) = -\frac{c}{\mu}z^{-2\beta-1}\frac{U(\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cX_0^{-2\beta})U(\frac{1+iu}{2\mu\beta}+1, 2+\frac{1}{2\beta}, cz^{-2\beta})}{U(\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cz^{-2\beta})^2}, \text{ for } \mu > 0$$

and

$$\hat{\Phi}(z,u) = -cz^{-2\beta-1}\left(\frac{2\beta+1}{1+iu} - \frac{1}{\mu}\right) \times \frac{U(1+\frac{1}{2\beta}-\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cX_0^{-2\beta})U(2+\frac{1}{2\beta}-\frac{1+iu}{2\mu\beta}, 2+\frac{1}{2\beta}, cz^{-2\beta})}{U(1+\frac{1}{2\beta}-\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cz^{-2\beta})^2},$$

for $\mu < 0$.

- Let us denote by $\tau_{X_0 \downarrow z} := \inf\{t \geq 0 : X_t = z\}$ the first time that the CEV process $(X_t)_{t \geq 0}$ starting at X_0 hits the level $0 < z < X_0$.

- By [Jeanblanc, Yor and Chesney 2009] the Laplace transform of$\tau_{X_0 \downarrow z}$ is given by

$$\mathrm{E}[e^{-s\tau_{X_0 \downarrow z}}] = \left(\frac{X_0}{z}\right)^{\beta + \frac{1}{2}} \exp\left(\frac{\epsilon}{2} c (X_0^{-2\beta} - z^{-2\beta})\right) \frac{M_{k,n}(cX_0^{-2\beta})}{M_{k,n}(cz^{-2\beta})}$$

with $\epsilon = \mathrm{sign}(\mu\beta)$, $n = \frac{1}{4\beta}$, $k = \epsilon\left(\frac{1}{2} + \frac{1}{4\beta}\right) - \frac{s}{2\beta|\mu|}$ and the Whittaker function

$$M_{k,n}(y) = y^{n+\frac{1}{2}} e^{-\frac{y}{2}} {}_1F_1(n - k + \frac{1}{2}, 2n+1, y),$$

where ${}_1F_1$ denotes the confluent hypergeometric function of the first kind with $\beta = \alpha - 1$ and $c = \dfrac{|\mu|}{\beta\sigma^2}$.

Theorem 9

Let $(X_t)_{0\leq t\leq T}$ denotes the CEV process solution to (29). Then $\inf_{t\in[0,T]} X_t$ has a continuous density on any compact set $K \subset (0, X_0)$, given by

$$z \in K \mapsto P_{CEV,\ Min}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(1+iu)T}\hat{\Psi}(z,u)du,$$

with

$$\hat{\Psi}(z,u) = \frac{cz^{-2\beta-1}}{\mu(1+\frac{1}{2\beta})}\frac{{}_1F_1(\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cX_0^{-2\beta})\,{}_1F_1(\frac{1+iu}{2\mu\beta}+1, 2+\frac{1}{2\beta}, cz^{-2\beta})}{{}_1F_1(\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cz^{-2\beta})^2}, \text{ for } \mu > 0$$

and

$$\hat{\Psi}(z,u) = cz^{-2\beta-1}\left(\frac{2\beta}{1+iu} - \frac{1}{\mu(1+\frac{1}{2\beta})}\right)$$
$$\times \frac{{}_1F_1(1+\frac{1}{2\beta}-\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cX_0^{-2\beta})\,{}_1F_1(2+\frac{1}{2\beta}-\frac{1+iu}{2\mu\beta}, 2+\frac{1}{2\beta}, cz^{-2\beta})}{{}_1F_1(1+\frac{1}{2\beta}-\frac{1+iu}{2\mu\beta}, 1+\frac{1}{2\beta}, cz^{-2\beta})^2}, \text{ for } \mu < 0.$$

Numerical testes

- we used our interpolated drift implicit scheme

$$\overline{Y}_t^n = \overline{Y}_{t_i}^n + (1-\alpha)\left(\mu \overline{Y}_{t_{i+1}}^n - \alpha \frac{\sigma^2}{2\overline{Y}_{t_{i+1}}^n}\right)(t-t_i) + \gamma(W_t - W_{t_i}),\ \text{for } t \in [t_i, t_{i+1}[, 0 \le i \le n$$

$Y_0 = {X_0}^{1-\alpha}$, and $\gamma = \sigma(1-\alpha)$.

- For n large enough, the positive solution to the above implicit scheme

is explicit and given by

$$\overline{Y}_{t_{i+1}}^n = \frac{\sqrt{2\sigma^2\alpha(\alpha-1)(1+\mu(\alpha-1)\frac{T}{n})\frac{T}{n} + (\gamma(W_{t_{i+1}} - W_{t_i}) + \overline{Y}_{t_i}^n)^2} + \gamma(W_{t_{i+1}} - W_{t_i}) + \overline{Y}_{t_i}^n}{2+2\mu(\alpha-1)\frac{T}{n}}.$$

- We choose $\alpha = 1.2$, $X_0 = 100$, $\mu = 0.1$, $\sigma = 0.2$, $T = 1$. The payoff function $g(x) = e^{-rT}(x^{\frac{1}{1-\alpha}} - K)_+$ is a discounted call function with $r = 0.1$. For the U-O option the strike is $K = 90$, and the barrier $\mathcal{D} = 150$. For the D-O option the strike is $K = 100$ and the barrier $\mathcal{U} = 90$.

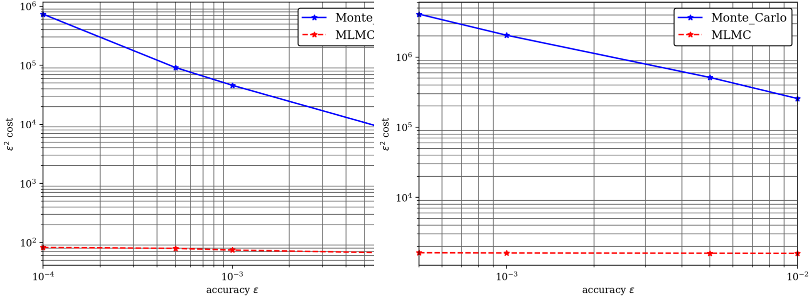
The tables and the figures below confirm the high performance of the improved MLMC.

Accuracy	Price	MLMC cost	MC cost	Saving
10^{-4}	3.0390	8.226×10^9	7.34×10^{13}	8922.33
5×10^{-4}	3.0391	3.17×10^8	3.67×10^{11}	1155.67
10^{-3}	3.041	7.436×10^7	4.587×10^{10}	616.91
10^{-2}	3.0452	6.539×10^5	5.734×10^7	87.69

Table: MLMC complexity tests for the U-O barrier option pricing of $\Pi_{\mathcal{D}}^{\text{U-O},X}$

Accuracy	Price	MLMC cost	MC cost	Saving
5×10^{-4}	11.102	6.483×10^9	1.642×10^{13}	2532.83
10^{-3}	11.103	1.608×10^9	2.053×10^{12}	1276.66
5×10^{-3}	11.106	6.379×10^7	2.053×10^{10}	321.77
10^{-2}	11.094	1.587×10^7	2.566×10^9	161.69

Table: MLMC complexity tests for the D-O barrier option pricing of $\Pi_{\mathcal{U}}^{\text{D-O},X}$



(a) Approximation of $\Pi_{\mathcal{D}}^{\text{U-O},X}$

(b) Approximation of $\Pi_{\mathcal{U}}^{\text{D-O},X}$

Figure: Comparison for the performances of MLMC vs classical MC algorithm under the CEV model

Outline

1. The one-dimensional setting with locally Lipschitz diffusion coefficient
2. The two-dimensional setting : Heston model

Log-Heston process

- For a stochastic volatility model of log-Heston $(X_t)_{t\in[0,T]}$ solution of the following SDE:

$$dX_t = (\mu - \frac{1}{2}V_t)dt + \sqrt{V_t} \ \ d\left(\rho W_t^v + \sqrt{1-\rho^2} W_t^s\right), \qquad (11)$$
$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^v.$$

with $\kappa,\ \theta,\ \sigma,\ X_0,\ V_0 > 0$, $\mu \in \mathbb{R}$, $\rho \in [-1,1]$, $T > 0$, and $(W_t^s)_{t\in[0,T]}$, $(W_t^v)_{t\in[0,T]}$ are two independent Brownian motions.

- Conditioned on V_u, the variance process V_t at time t, $t \geq u$

$$V_t \stackrel{d}{=} \frac{\sigma^2(1-\exp\left(-\kappa(t-u)\right))}{4\kappa}\chi_d^2\Big(\frac{4\kappa\exp\left(-\kappa(t-u)\right)}{\sigma^2(1-\exp\left(-\kappa(t-u)\right))}V_u\Big),$$

where $\chi_d^2(\lambda)$ denotes a non central chi-squared random variable with $d := \frac{4\theta\kappa}{\sigma^2}$ degrees of freedom and non-centrality parameter λ.

Log-Heston process

- For $0 \leq u \leq t \leq T$ we can rewrite (11) as follows

$$X_t = X_u + \mu(t-u) - \frac{1}{2}\int_u^t V_s ds + \frac{\rho}{\sigma}\left(V_t - V_u - \kappa\theta(t-u) + \kappa\int_u^t V_s ds\right)$$
$$+ \sqrt{1-\rho^2}\int_u^t \sqrt{V_s} dW_s^s,$$

since

$$\int_u^t \sqrt{V_s} dW_s^v = \frac{1}{\sigma}\left(V_t - V_u - \kappa\theta(t-u) + \kappa\int_u^t V_s ds\right).$$

- In what follows, we assume that the Feller condition $2\kappa\theta > \sigma^2$ holds true to guarantee the positivity of the CIR process.

Semi-exact log-Heston scheme

- For $t_i = \frac{iT}{n}$, $0 \leq i \leq n$, we consider the Semi-exact log-Heston discrete scheme

$$
\begin{aligned}
\overline{X}^n_{t_{i+1}} &= \overline{X}^n_{t_i} + (\mu - \frac{1}{2} V_{t_i})(t_{i+1} - t_i) \\
&\quad + \frac{\rho}{\sigma}\Big(V_{t_{i+1}} - V_{t_i} - \kappa\theta(t_{i+1} - t_i) + \kappa V_{t_i}(t_{i+1} - t_i) \Big) \\
&\quad + \sqrt{1-\rho^2}\sqrt{V_{t_i}}(W^s_{t_{i+1}} - W^s_{t_i}),
\end{aligned}
$$

- We introduce the following interpolated Semi-exact log-Heston scheme

$$
\begin{aligned}
\overline{X}^n_t &= \overline{X}^n_{t_i} + (\mu - \frac{1}{2} V_{t_i})(t - t_i) \\
&\quad + \frac{\rho}{\sigma}\Big(\frac{n}{T}(V_{t_{i+1}} - V_{t_i})(t - t_i) - \kappa\theta(t - t_i) + \kappa V_{t_i}(t - t_i) \Big) \\
&\quad + \sqrt{1-\rho^2}\sqrt{V_{t_i}}(W^s_t - W^s_{t_i}).
\end{aligned}
$$

- The main advantages of this Brownian interpolation is that it enables straightforward use of the Brownian bridge technique for pricing Barrier options.

Strong convergence rate

Theorem 10

Let $T > 0$. For all $p > 1$, there exists a constant C_p such that

$$\max_{1 \le i \le n} \left(\mathbb{E} |X_{t_i} - \overline{X}^n_{t_i}|^p \right)^{\frac{1}{p}} \le C_p \left(\frac{T}{n} \right)^{1/2}$$

- The above barrier option prices can be approximated by

$$\overline{\pi}_{\mathcal{D}} := \mathbb{E}\Big[f(\overline{X}^n_T)\prod_{i=0}^{n-1} 1_{\{\inf_{t\in[t_i,t_{i+1}]}\overline{X}^n_t>\mathcal{D}\}}\Big] \text{ and } \overline{\pi}_{\mathcal{U}} := \mathbb{E}\Big[f(\overline{X}^n_T)\prod_{i=0}^{n-1} 1_{\{\sup_{t\in[t_i,t_{i+1}]}\overline{X}^n_t<\mathcal{U}\}}\Big].$$

- To get more accurate approximations, we use the Brownian bridge technique. For $x \in \mathbb{R}$, $(x)_+$ stands for $\max(x, 0)$.

Proposition 2

Under the above notations, for $h = \frac{T}{n}$, we have

$$\overline{\pi}_U = \mathbb{E}\Big[f(\overline{X}^n_T)\prod_{i=0}^{n-1}(1-\overline{p}_i)\Big], \text{ where } \overline{p}_i = \exp\Big(\frac{-2(U-\overline{X}^n_{t_i})_+(U-\overline{X}^n_{t_{i+1}})_+}{(1-\rho^2)V_{t_i}h}\Big),$$

and

$$\overline{\pi}_D = \mathbb{E}\Big[f(\overline{X}^n_T)\prod_{i=0}^{n-1}(1-\overline{q}_i)\Big], \text{ where } \overline{q}_i := \exp\Big(\frac{-2(\overline{X}^n_{t_i}-D)_+(\overline{X}^n_{t_{i+1}}-D)_+}{(1-\rho^2)V_{t_i}h}\Big).$$

Brownian bridge for the MLMC method

- Thus, the improved MLMC method approximates $\overline{\pi}_{\mathcal{U}}$ by

$$\bar{P}_U := \frac{1}{N_0}\sum_{k=1}^{N_0}\overline{P}^f_{0,k} + \sum_{\ell=1}^{L}\frac{1}{N_\ell}\sum_{k=1}^{N_\ell}\left(\overline{P}^f_{\ell,k} - \overline{P}^c_{\ell-1,k}\right),$$

$$\overline{P}^f_\ell : = f(\overline{X}^{2^\ell}_T)\prod_{i=0}^{2^\ell-1}(1-\overline{p}^{2^\ell}_i) \text{ with } \overline{p}^{2^\ell}_i = \exp\left(\frac{-2(U-\overline{X}^{2^\ell}_{t_i})_+(U-\overline{X}^{2^\ell}_{t_{i+1}})_+}{(1-\rho^2)V_{t^\ell_i}h_\ell}\right),$$

$$\overline{P}^c_{\ell-1} := f(\overline{X}^{2^{\ell-1}}_T)\prod_{i=0}^{2^\ell-1}(1-\overline{p}^{2^{\ell-1}}_i) \text{ with } \overline{p}^{2^{\ell-1}}_i = \exp\left(\frac{-2(U-\overline{X}^{2^{\ell-1}}_{t^\ell_i})_+(U-\overline{X}^{2^{\ell-1}}_{t^\ell_{i+1}})_+}{(1-\rho^2)V_{t^{\ell-1}_i}h_\ell}\right),$$

where the condition $\mathbb{E}\left[\overline{P}^f_{\ell-1}\right] = \mathbb{E}\left[\overline{P}^c_{\ell-1}\right]$ is satisfied.

Brownian bridge for the MLMC method (continued)

- Similarly, the improved MLMC method approximates $\bar{\pi}_{\mathcal{D}}$ by

$$\bar{Q}_D := \frac{1}{N_0}\sum_{k=1}^{N_0}\overline{Q}^f_{0,k} + \sum_{\ell=1}^{L}\frac{1}{N_\ell}\sum_{k=1}^{N_\ell}\left(\overline{Q}^f_{\ell,k} - \overline{Q}^c_{\ell-1,k}\right), \tag{12}$$

$$\overline{Q}^f_\ell := f(\overline{X}^{2^\ell}_T)\prod_{i=0}^{2^\ell-1}(1-\overline{q}^{2^\ell}_i) \text{ with } \overline{q}^{2^\ell}_i = \exp\left(\frac{-2(\overline{X}^{2^\ell}_{t^\ell_i} - D)_+(\overline{X}^{2^\ell}_{t^\ell_{i+1}} - D)_+}{(1-\rho^2)V_{t^\ell_i}h_\ell}\right),$$

$$\overline{Q}^c_{\ell-1} := f(\overline{X}^{2^{\ell-1}}_T)\prod_{i=0}^{2^\ell-1}(1-\overline{q}^{2^{\ell-1}}_i) \text{ with } \overline{q}^{2^{\ell-1}}_i = \exp\left(\frac{-2(\overline{X}^{2^{\ell-1}}_{t^\ell_i} - D)_+(\overline{X}^{2^{\ell-1}}_{t^\ell_{i+1}} - D)_+}{(1-\rho^2)V_{t^{\ell-1}_i}h_\ell}\right).$$

Extreme path events

Lemma 11

For any $\eta > 0$ *and* $h_\ell = 2^{-\ell} T$*, the following extreme path events satisfy*

$$\max_{0 \le i \le 2^\ell} \mathbb{P}\left(\max\left(|\overline{X}^{2^\ell}_{t_i^\ell}|, |\overline{X}^{2^{\ell-1}}_{t_i^\ell}| \right) > h_\ell^{-\eta} \right) = o(h_\ell^q)$$

$$\max_{0 \le i \le 2^\ell} \mathbb{P}\left(\max\left((|X_{t_i^\ell} - \overline{X}^{2^\ell}_{t_i^\ell}|, |X_{t_i^\ell} - \overline{X}^{2^{\ell-1}}_{t_i^\ell}|, |\overline{X}^{2^\ell}_{t_i^\ell} - \overline{X}^{2^{\ell-1}}_{t_i^\ell}|) \right) > h_\ell^{1/2-\eta} \right) = o(h_\ell^q)$$

$$\mathbb{P}\left(\sup_{t \in [0,T]} |W_t - W_{t_i^\ell}| > h_\ell^{\frac{1}{2}-\eta} \right) = o(h_\ell^q)$$

for all $q > 0$.

Lemma 12

Let $2\kappa\theta > \sigma^2$*, for any* $\eta > 0$ *and* $h_\ell = 2^{-\ell}T$*, the following extreme path events satisfy*

$$\max_{0\le i\le 2^\ell} \mathbb{P}\left(\max\left((|\frac{1}{V_{t_i^{\ell-1}}}|,|\frac{1}{V_{t_i^\ell}}|) > h_\ell^{-\eta}\right)\right) = o(h_\ell^q), \quad \textit{for all } 0 < q < \frac{2\kappa\theta}{\sigma^2}\eta$$

$$\max_{0\le i\le 2^\ell} \mathbb{P}\left((|\frac{1}{V_{t_i^\ell}} - \frac{1}{V_{t_i^{\ell-1}}}|) > h_\ell^{1/2-\eta}\right) = o(h_\ell^q), \quad \textit{for all } 0 < q < \frac{\kappa\theta}{\sigma^2}\eta$$

$$\max_{0\le i\le 2^\ell} \mathbb{P}\left(\int_{t_i^\ell}^{t_{i+1}^\ell} V_s ds > h_\ell^{1-\eta}\right) = o(h_\ell^q), \quad \textit{for all } q > 0$$

$$\max_{0\le i\le 2^\ell} \mathbb{P}\left(\sup_{t\in[t_i^\ell,t_{i+1}^\ell]} |\int_{t_i^\ell}^{t} \sqrt{V_s} dW_s^s| > h_\ell^{\frac{1}{2}-\eta}\right) = o(h_\ell^q), \quad \textit{for all } q > 0.$$

MLMC variance and complexity analysis

- Let us assume that there exists a positive constant $a > \frac{1}{2}$ such that $\kappa\theta/\sigma^2 > a$ so that the Feller's condition is satisfied.

- We need the following assumptions. Let f denote a payoff function satisfying: $\exists C_1 > 0$ s.t. $\forall t \geq 0, x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq C_1|x - y|.$$

This implies $\exists C_2 > 0$ s.t. $\forall x \in \mathbb{R}$,

$$|f(x)| \leq C_2(1 + |x|).$$

- Provided that $\inf_{t\in[0,T]} X_t$ has a bounded density in the neighborhood of the barrier $\mathcal{D}$, we get

$$\mathrm{Var}(\overline{Q}_\ell^f - \overline{Q}_\ell^c) = O(h_\ell^{\frac{1}{2}-\delta}) \text{ for } \delta \in (\frac{3}{2(a+3)}, \frac{1}{2}), \quad \text{with} \quad a > \frac{1}{2}.$$

Antithetic MLMC method for Heston model

- We define the Heston model solution to

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{V_t} S_t d\tilde{W}^s_t \\
dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW^v_t
\end{aligned}
\tag{13}
$$

where $\tilde{W}^s$ and W^v are two correlated B.m. with correlation ρ.

- For $t_i = \frac{iT}{n}$, $0 \le i \le n$, we consider the following approximate scheme [Giles; Szpruch, 2013]:

$$
\begin{aligned}
\hat{S}^n_{t_{i+1}} &= \hat{S}^n_{t_i}\Big(1 + \mu h + \sqrt{\hat{V}_{t_i}}(\tilde{W}^s_{t_{i+1}} - \tilde{W}^s_{t_i}) + \frac{1}{2}\hat{V}_{t_i}\left((\tilde{W}^s_{t_{i+1}} - \tilde{W}^s_{t_i})^2 - h\right) \\
&\quad + \frac{1}{4}\sigma\left((\tilde{W}^s_{t_{i+1}} - \tilde{W}^s_{t_i})(W^v_{t_{i+1}} - W^v_{t_i}) - \rho h\right)\Big), \\
\hat{V}_{t_{i+1}} &= \hat{V}_{t_i} + \kappa(\theta - V_{t_{i+1}})h + \sigma\sqrt{\hat{V}_{t_i}}(W^v_{t_{i+1}} - W^v_{t_i}) + \frac{1}{2}\sigma^4\left((W^v_{t_{i+1}} - W^v_{t_i})^2 - h\right).
\end{aligned}
$$

Where $\hat{S}^n_{t_i}$ is the truncated Milstein scheme with step $h = \frac{T}{n}$.

Antithetic MLMC method

- Thus, the improved MLMC method approximates $\bar{\pi}_D$ by

$$\hat{Q}_D := \frac{1}{N_0}\sum_{k=1}^{N_0}\hat{Q}^f_{0,k} + \sum_{\ell=1}^{L}\frac{1}{N_\ell}\sum_{k=1}^{N_\ell}\Big(\frac{1}{2}(\overline{\hat{Q}}^f_{\ell,k} + \hat{Q}^a_{\ell,k}) - \hat{Q}^c_{\ell-1,k}\Big),$$

where $(\hat{Q}^f_{\ell,k})_{1\le k\le N_\ell}$ and $(\hat{Q}^c_{\ell-1,k})_{1\le k\le N_\ell}$ are respectively independent copies of $\hat{Q}^f_\ell$ and $\hat{Q}^c_{\ell-1}$ given by

$$\hat{Q}^f_\ell := f(\hat{S}^{2^\ell}_T)\prod_{i=0}^{2^\ell-1}(1-\hat{q}^{2^\ell}_i) \text{ with } \hat{q}^{2^\ell}_i = \exp\Bigg(\frac{-2(\hat{S}^{2^\ell}_{t^\ell_i} - D)_+(\hat{S}^{2^\ell}_{t^\ell_{i+1}} - D)_+}{(1-\rho^2)\hat{V}_{t^\ell_i}h_\ell}\Bigg)$$

$$\hat{Q}^c_{\ell-1} := f(\hat{S}^{2^{\ell-1}}_T)\prod_{i=0}^{2^\ell-1}(1-\hat{q}^{2^{\ell-1}}_i) \text{ with } \hat{q}^{2^{\ell-1}}_i = \exp\Bigg(\frac{-2(\hat{S}^{2^{\ell-1}}_{t^\ell_i} - D)_+(\hat{S}^{2^{\ell-1}}_{t^\ell_{i+1}} - D)_+}{(1-\rho^2)\hat{V}_{t^{\ell-1}_i}h_\ell}\Bigg).$$

- we consider a D-O Call option with strike $K = 100$ and barrier $B = 75$. We suppose that the stock price is following a Heston model (13) with parameters $S_0 = 100$, $V_0 = 0.1$, $\sigma = 0.1$, $\theta = 0.1$, $\kappa = 2$, $\rho = -0.5$ and $\mu = -0.02$.
- The performance of the improved MLMC and Antithetic MLMC is given in the table and figure below.

Estimation	α	β	MLMC Complexity
MLMC Semi Exact	1.173726	0.962958	$\varepsilon^{2.03}$
Antithetic MLMC	0.724667	0.662456	$\varepsilon^{2.47}$

Table: Time complexity of MLMC Semi xact Vs antithetic MLMC methods

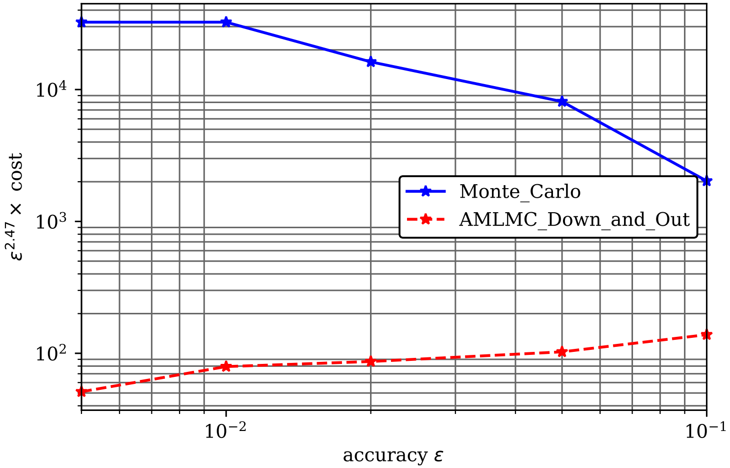


Figure: Performance of the Antithetic MLMC algorithm under the Heston model for pricing Down & Out Barrir options

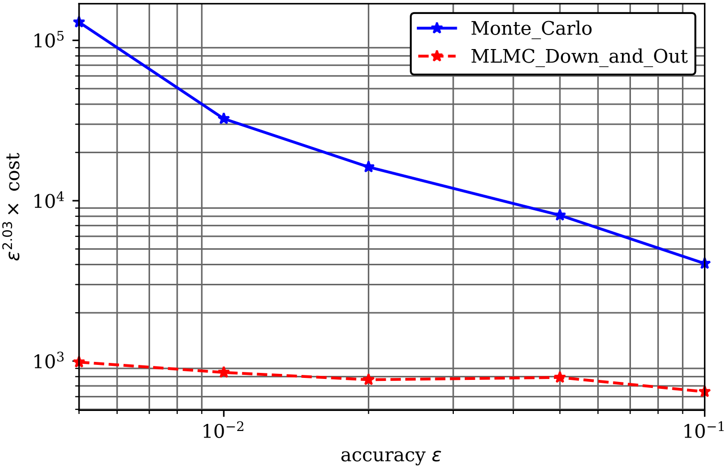


Figure: Performance of the MLMC Semi Exact algorithm under the Heston model for pricing Down & Out Barrir options

- Ben Derouich, M. and Kebaier, A: The interpolated drift implicit Euler scheme Multilevel Monte Carlo method for pricing Barrier options and applications to the CIR and CEV models https://arxiv.org/abs/2210.00779 (2022).

Thank you !