Estimation of a stable Cox-Ingersoll-Ross process from high-frequency observations

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Outline

1. Stable CIR process : properties and moments estimates
2. Joint estimation of the pure-jump stable CIR process from high-frequency observations
3. Volatility and jump activity estimation in presence of a Brownian component

Stable CIR process - Continuous Branching process with Immigration

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dB_t + \delta X_{t-}^{1/\alpha} dL_t^\alpha, \quad t \geq 0 \quad x_0 \geq 0$$

- (L_t^α) non-symmetric pure-jump Lévy process, strictly α-stable, with triplet $(0, 0, F^\alpha)$ and representation

$$L_t^\alpha = \int_0^t \int z\tilde{\mu}(ds, dz), \qquad \mathbb{E}e^{iuL_1^\alpha} = e^{-|u|^\alpha(1-\text{signe}(u)\tan(\pi\alpha/2))}$$

  where $\tilde{\mu} = \mu - \overline{\mu}$ is a compensated Poisson random measure with compensator

$$\overline{\mu}(dt, dz) = dtF^\alpha(dz)$$

  where the Lévy measure is given by

$$F^\alpha(dz) = \frac{c_\alpha}{z^{1+\alpha}} 1_{(0,+\infty)}(z)dz, \quad 1 < \alpha < 2$$

- (B_t) standard Brownian motion independent of (L_t^α)

Laplace transform

$$\mathbb{E}(e^{-uX_t}) = \exp\left(-x_0 v_t(u) - \int_{v_t(u)}^{u} \frac{F(z)}{R(z)} dz\right),$$

where $t \to v_t(u)$ is the unique locally bounded solution of

$$\frac{\partial}{\partial t} v_t(u) = -R(v_t(u)), \quad v_0(u) = u. \tag{1}$$

$$R(z) = \frac{\sigma^2}{2} z^2 + \frac{\overline{\delta}^{\alpha}}{\alpha} z^{\alpha} + bz, \quad F(z) = az,$$

with $\overline{\delta} = \delta(\alpha/|\cos(\frac{\pi\alpha}{2})|)^{1/\alpha}$.

- $\sigma = 0$ explicit expression of $v_t(u)$

$$\text{if } b \neq 0 \quad v_t(u) = \frac{ue^{-bt}}{\left(1 + \frac{\overline{\delta}^{\alpha}}{\alpha b} u^{\alpha-1}\left(1 - e^{-(\alpha-1)bt}\right)\right)^{\frac{1}{\alpha-1}}},$$

$$\text{if } b = 0 \quad v_t(u) = u\left(\frac{\alpha}{\alpha + (\alpha-1)\overline{\delta}^{\alpha} u^{\alpha-1} t}\right)^{\frac{1}{\alpha-1}}.$$

- $\sigma \neq 0 \quad \delta \neq 0$ not explicit

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dB_t + \delta X_{t-}^{1/\alpha} dL_t^{\alpha}, \quad t \geq 0 \quad x_0 \geq 0$$

- Existence and uniqueness of a strong solution such that

$$\mathbb{P}(X_t \geq 0, \ \forall t \geq 0) = 1$$

  if $a \geq 0, \ b \in \mathbb{R}, \ \sigma \geq 0, \ \delta \geq 0$

- The solution satisfies

$$\mathbb{P}(X_t > 0, \ \forall t \geq 0) = 1$$

  if $x_0 > 0, \ 2a \geq \sigma^2 > 0, \ b \in \mathbb{R}, \ \delta \geq 0$

  For the pure-jump stable CIR process ($\sigma = 0$)

$$\mathbb{P}(X_t > 0, \ \forall t \geq 0) = 1$$

  if $a > 0, \ b \in \mathbb{R}, \ \delta > 0$

  Kawazu-Watanabe (71), Fu-Li (10), Jiao-Ma-Scotti (18)

Moment estimates

Proposition *Bayraktar-C. (24)*

We have

1. $\forall p \in (0, \alpha)$
$$\mathbb{E}_{|\mathcal{F}_s}\left(\sup_{s \le u \le t} X_u^p\right) \le C_p(1 + X_s^p)$$
2. if $\sigma = 0$, $\delta > 0$, $a > 0$, $\forall p > 0$
$$\sup_{t \in [0,1]} \mathbb{E}\left(\frac{1}{X_t^p}\right) < +\infty$$
3. if $\sigma > 0$, $\delta > 0$, $2a > \sigma^2$, $\forall 1 \le p < 2a/\sigma^2$
$$\sup_{t \ge 0} \mathbb{E}\left(\frac{1}{X_t^p}\right) < +\infty$$

2. 3. proof based on the expression of the Laplace transform of a CBI process

Parametric estimation : recent results for the stable CIR process

- Small noise and pure-jump stable process *Ma-Yang (14), Yang (17)*

$$dX_t = (a - bX_t)dt + \varepsilon\delta X_{t-}^{1/\alpha} dL_t^\alpha, \quad t \geq 0, \quad x_0 > 0$$

  $\alpha \in (1, 2)$ known
  Observations : $(X_{i\Delta_n})_{i\in\{1,\dots,n\}}$, $\Delta_n \to 0$, $n\Delta_n = T$ fixed, $(T = 1)$
  Estimation of (a, b, δ) by approximating the likelihood function (depends on α)

- Fixed step-size observations, long-time behavior *Li-Ma (15)*

$$dX_t = (a - bX_t)dt + \delta X_{t-}^{1/\alpha} dL_t^\alpha, \quad t \geq 0, \quad x_0 > 0$$

  Observations : $(X_{i\Delta})_{i\in\{1,\dots,n\}}$, Δ fixed and $n \to \infty$
  Estimation of (a, b) assuming $b > 0$, (X_t) is geometrically ergodic
  Explicit least squares estimators of (a, b) (independent of α and δ) based on the equation satisfied by $e^{bt}X_t$

- Continuous time observations with $\sigma > 0$ *Barczy-Ben Alaya-Kebaier-Pap (19)*

$$dX_t = (a - bX_t)dt + \sigma dB_t + \delta X_{t-}^{1/\alpha} dL_t^{\alpha}, \quad t \geq 0, \quad x_0 > 0$$

Observations : $(X_t)_{t \in [0,T]}$, $T \to \infty$

$a \geq 0$, $\sigma > 0$, $\delta > 0$ and $\alpha \in (1,2)$ are known

Estimation of b

Explicit expression of the maximum likelihood estimator $\hat{b}$ from Girsanov's Theorem (depends on $a \geq 0$, $\delta > 0$ and α)

  - $b > 0$: consistency and asymptotic normality with rate $\sqrt{T}$
  - $b = 0$: consistency
  - $b < 0$: consistency and asymptotic mixed normality with rate e^{-bT}

Estimation of a pure-jump stable CIR process

In this part

$$dX_t = (a - bX_t)dt + \delta X_{t-}^{1/\alpha} dL_t^{\alpha}, \quad t \geq 0, \quad x_0 > 0$$

- Estimation of $\theta = (a, b, \delta, \alpha)$
  $(X_t)_{t \in [0,1]}$ solves the stochastic equation for the parameter value $\theta_0 \in (0, \infty) \times \mathbb{R} \times (0, \infty) \times (1, 2) = \Theta$
- High-frequency observations : $(X_{\frac{i}{n}})_{i \in \{0, \ldots, n\}}$
- Estimating functions method based on an approximation of the conditional distribution of $X_{\frac{i}{n}}$ given $X_{\frac{i-1}{n}}$

  C.-Gloter (19) (20) : SDE with Lipschitz coefficients, driven by symmetric locally stable process

Pure jump stable CIR process : estimation method

- Euler approximation

$$X_{\frac{i}{n}} \backsimeq X_{\frac{i-1}{n}} + (a - bX_{\frac{i-1}{n}})\frac{1}{n} + \delta X_{\frac{i-1}{n}}^{1/\alpha}(L^{\alpha}_{\frac{i}{n}} - L^{\alpha}_{\frac{i-1}{n}})$$

$$L^{\alpha}_{\frac{i}{n}} - L^{\alpha}_{\frac{i-1}{n}} \text{ has the distribution of } \frac{1}{n^{1/\alpha}} L^{\alpha}_1$$

- Approximation of the log-likelihood function

$$L_n(\theta) = \sum_{i=1}^{n} \log \left[\frac{n^{1/\alpha}}{\delta X_{\frac{i-1}{n}}^{1/\alpha}} \varphi_\alpha \left(n^{1/\alpha} \frac{X_{\frac{i}{n}} - X_{\frac{i-1}{n}} - \frac{a}{n} + \frac{b}{n} X_{\frac{i-1}{n}}}{\delta X_{\frac{i-1}{n}}^{1/\alpha}} \right) \right]$$

where φ_α is the density of L^{α}_1

- Estimator

$$\hat{\theta}_n \text{ solution of } \nabla_\theta L_n(\theta) = 0 \text{ on } \Theta$$

Estimation results

We define $G_n(\theta) = -\nabla_\theta L_n(\theta)$ (recall that L_n is an approximation of the log-likelihood function)
and $J_n(\theta) = \nabla_\theta G_n(\theta)$ (approximation of the information)

Let u_n be the non-diagonal rate

$$u_n = \begin{pmatrix} \frac{1}{n^{1/\alpha_0 - 1/2}} Id & 0 \\ 0 & \frac{1}{\sqrt{n}} v_n \end{pmatrix}, \quad v_n = \begin{pmatrix} \delta_0 & -\delta_0 \log(n)/\alpha_0 \\ 0 & 1 \end{pmatrix}$$

Proposition *Bayraktar-C. (24)*

1. $$\sup_{\theta \in W_n^{(\eta)}} \log(n)^q || u_n^T J_n(\theta) u_n - I(\theta_0) || \to 0$$
2. $u_n^T G_n(\theta_0)$ stably converges in law to $I(\theta_0)^{1/2} \mathcal{N}$

where $I(\theta_0)$ is a symmetric non-negative and non-singular matrix depending on $(X_t)_{t \in [0,1]}$ and $\mathcal{N}$ is a standard Gaussian variable independent of $I(\theta_0)$

- From the previous convergences, we obtain the following result

Theorem *Bayraktar-C. (24)*

There exists a sequence $(\hat{\theta}_n)$, such that $\lim_n \mathbb{P}(G_n(\hat{\theta}_n) = 0) = 1$, that converges in probability to θ_0 .

Moreover we have the stable convergence in law with respect to $\sigma(L_s^{\alpha_0}, s \leq 1)$

$$u_n^{-1}\left(\hat{\theta}_n - \theta_0\right) \xrightarrow{\mathcal{L}_s} I(\theta_0)^{-1/2}\mathcal{N},$$

where $\mathcal{N}$ is a standard Gaussian variable independent of $I(\theta_0)$

- If α_0 is known (or δ_0 known), we obtain a similar result in estimating (a, b, δ) (or (a, b, α)) with the diagonal rate

$$u_n = \begin{pmatrix} \frac{1}{n^{1/\alpha_0 - 1/2}} Id & 0 \\ 0 & \frac{1}{\sqrt{n}} \end{pmatrix}, \quad u_n = \begin{pmatrix} \frac{1}{n^{1/\alpha_0 - 1/2}} Id & 0 \\ 0 & \frac{1}{\log n \sqrt{n}} \end{pmatrix}$$

and non-singular information

Information for the estimation of (a, δ, α)

$$I(a,\delta,\alpha) = \begin{pmatrix} \frac{1}{\delta^2}\mathbb{E}h_\alpha^2(L_1^\alpha)\int_0^1 \frac{ds}{X_s^{1/\alpha}} & Sym. \\ I^{2,1} & I^{2,2} \end{pmatrix}$$

$$I^{2,1} = \begin{pmatrix} \frac{1}{\delta}\mathbb{E}(h_\alpha k_\alpha)(L_1^\alpha)\int_0^1 \frac{ds}{X_s^{1/\alpha}} \\ -\frac{1}{\delta\alpha^2}\mathbb{E}(h_\alpha k_\alpha)(L_1^\alpha)\int_0^1 \frac{\log(X_s)}{X_s^{1/\alpha}}ds - \frac{1}{\delta}\mathbb{E}(f_\alpha h_\alpha)(L_1^\alpha)\int_0^1 \frac{ds}{X_s^{1/\alpha}} \end{pmatrix}$$

$$I^{2,2} = \begin{pmatrix} \mathbb{E}k_\alpha^2(L_1^\alpha) & -\frac{1}{\alpha^2}\mathbb{E}k_\alpha^2(L_1^\alpha)\int_0^1 \log(X_s)ds - \mathbb{E}(k_\alpha f_\alpha)(L_1^\alpha) \\ Sym. & \mathbb{E}f_\alpha^2(L_1^\alpha) + \frac{1}{\alpha^4}\mathbb{E}k_\alpha^2(L_1^\alpha)\int_0^1 \log(X_s)^2 ds + \frac{2}{\alpha^2}\mathbb{E}(f_\alpha k_\alpha)(L_1^\alpha)\int_0^1 \log(X_s)ds \end{pmatrix}$$

with $h_\alpha = \varphi'_\alpha/\varphi_\alpha$, $k_\alpha(z) = 1 + zh_\alpha(z)$, $f_\alpha = \partial_\alpha\varphi_\alpha/\varphi_\alpha$.

Some partial results on efficiency in high-frequency setting : a toy model

$$X_t = x_0 + at + \delta S_t^\alpha$$

S_t^α : symmetric α-stable process
observations $X_{\frac{i}{n}}$ $0 \leq i \leq n$, the log-likelihood function is explicit
Brouste-Masuda (18)

- LAN property with non-diagonal rate u_n in estimating $\theta = (a, \delta, \alpha)$ and non-singular information $\bar{I}$

$$u_n = \begin{pmatrix} \frac{1}{n^{1/\alpha - 1/2}} & 0 \\ 0 & \frac{1}{\sqrt{n}} v_n \end{pmatrix}, \quad v_n = \begin{pmatrix} \delta_0 & -\delta_0 \log(n)/\alpha_0 \\ 0 & 1 \end{pmatrix}$$

$$\bar{I}(a, \delta, \alpha) = \begin{pmatrix} \frac{1}{\delta^2}\mathbb{E}h_\alpha^2(S_1^\alpha) & 0 & 0 \\ 0 & \mathbb{E}k_\alpha^2(S_1^\alpha) & -\mathbb{E}(k_\alpha f_\alpha)(S_1^\alpha) \\ 0 & -\mathbb{E}(k_\alpha f_\alpha)(S_1^\alpha) & \mathbb{E}f_\alpha^2(S_1^\alpha) \end{pmatrix}$$

with $h_\alpha = \varphi'_\alpha / \varphi_\alpha$, $k_\alpha(z) = 1 + zh_\alpha(z)$ and $f_\alpha = \partial_\alpha \varphi_\alpha / \varphi_\alpha$.
(extension to L_t^α, the information is not bloc diagonal)

Drift estimation

- The previous result states existence of a consistent estimator with optimal rate of convergence but there might be other sequences that solve the estimating equation that are not consistent.
- Uniqueness is obtained if δ_0 and α_0 are known or if we have preliminary estimators $\tilde{\delta}_n$ and $\tilde{\alpha}_n$.

Theorem *Bayraktar-C. (24)*

Let $G_{n,d}$ be the approximation of the score function restricted to the drift parameters (a, b)

$$G_{n,d}(a,b) = \nabla_{(a,b)} L_n(a, b, \tilde{\delta}_n, \tilde{\alpha}_n)$$

We assume that $(a_0, b_0) \in \text{Int}(\Theta)$ for a compact set $\Theta \subset \mathbb{R}_+ \times \mathbb{R}$ and that $\frac{\sqrt{n}}{\log(n)^p}(\tilde{\delta}_n - \delta_0)$ and $\frac{\sqrt{n}}{\log(n)^p}(\tilde{\alpha}_n - \alpha_0)$ are tight, for $p > 0$.
Then any sequence $(\tilde{a}_n, \tilde{b}_n)$ that solves $G_{n,d}(\tilde{a}_n, \tilde{b}_n) = 0$ converges in probability to (a_0, b_0) and this sequence is unique.

Moreover the sequence $\frac{n^{1/\alpha_0}}{\sqrt{n}\log(n)^p}(\tilde{a}_n - a_0, \tilde{b}_n - b_0)$ is tight.

Preliminary estimators $(\tilde{\delta}_n, \tilde{\alpha}_n)$

Non parametric methods for semimartingales based on p-order power variation (*Todorov-Tauchen (11), Todorov (13), Todorov (15)*)
The first and two order power variation are defined by

$$V_n^1(p,X) = \sum_{i=2}^{n} |\Delta_i^n X - \Delta_{i-1}^n X|^p, \quad V_n^2(p,X) = \sum_{i=4}^{n} |\Delta_i^n X - \Delta_{i-1}^n X + \Delta_{i-2}^n X - \Delta_{i-3}^n X)|^p$$

with $\Delta_i^n X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}$.

Theorem *Todorov (13)*

1. For $p \in (0, \alpha)$, we have the convergences in probability

$$\frac{n^{p/\alpha}}{n} V_n^1(p,X) \to C_0(\alpha,\delta), \qquad \frac{n^{p/\alpha}}{n} V_n^2(p,X) \to 2^{p/\alpha} C_0(\alpha,\delta)$$

2.
$$\tilde{\alpha}_n = \frac{p \log 2}{\log(V_n^2(p,X)/V_n^1(p,X))} 1_{V_n^1(p,X) \neq V_n^2(p,X)}$$

$\sqrt{n}(\tilde{\alpha}_n - \alpha)$ stably converges in law, for $p \in (\frac{|\alpha-1|}{2(\alpha \wedge 1)}, \alpha/2)$ and $\alpha > 2/3$.

We can apply the previous result to the stable CIR process X wit $p = 1/2$ since $\alpha_0 > 1$.
We obtain the preliminary estimator $\tilde{\alpha}_n$ which is consistent with rate of convergence $\sqrt{n}$.

Next, we estimate δ by

$$\tilde{\delta}_n = \left(\frac{1}{m_p(\tilde{\alpha}_n)} \frac{n^{p/\tilde{\alpha}_n}}{n} \sum_{i=2}^{n} \left| \frac{\Delta_i^n X - \Delta_{i-1}^n X}{X_{\frac{i-1}{n}}^{1/\tilde{\alpha}_n}} \right|^p \right)^{1/p}$$

where $m_p(\alpha) = \mathbb{E}|L_1^\alpha - \overline{L}_1^\alpha|^p = \mathbb{E}|S_1^\alpha|^p$ where S_1^α is symmetric α-stable

For $p = 1/2$, $\frac{\sqrt{n}}{\log(n)}(\tilde{\delta}_n - \delta_0)$ is tight

One-step improvement - implementation

- Existence but not uniqueness result for the joint estimation
  - There exists a joint estimator $\hat{\theta}_n = (\hat{a}_n, \hat{b}_n, \hat{\delta}_n, \hat{\alpha}_n)$ consistent, rate optimal and probably efficient that solves $G_n(\hat{\theta}_n) = 0$.
- Preliminary estimators $\tilde{\delta}_n$ and $\tilde{\alpha}_n$
  - Very easy to implement, consistent not efficient
- Preliminary estimators $\tilde{a}_n$ and $\tilde{b}_n$
  - Obtained by solving $\nabla_{(a,b)} L_n(a, b, \tilde{\delta}_n, \tilde{\alpha}_n) = 0$ (unique solution)
  - Consistent but not rate optimal
- One-step improvement or k-step improvement
  - Use the theoretical properties of $\hat{\theta}_n$ to improve the asymptotic properties of the preliminary estimator $\tilde{\theta}_n$

- One-step improvement : start with $\hat{\theta}_{0,n} = \tilde{\theta}_n$ and next

$$\hat{\theta}_{1,n} = \hat{\theta}_{0,n} - \nabla_\theta G_n(\hat{\theta}_{0,n})^{-1} G_n(\hat{\theta}_{0,n}),$$

  Requires the computation of the density φ_α of the non-symmetric stable variable L_1^α (as well as its derivatives)
- We can prove $u_n^{-1}(\hat{\theta}_{1,n} - \hat{\theta}_n) = o(1)$

Then

$$u_n^{-1}\left(\hat{\theta}_{1,n} - \theta_0\right) = u_n^{-1}\left(\hat{\theta}_n - \theta_0\right) + o(1)$$

Corollary

We deduce that $\hat{\theta}_{1,n}$ inherits the asymptotic properties of $\hat{\theta}_n$:
we have the stable convergence in law with respect to $\sigma(L_s^{\alpha_0}, s \leq 1)$

$$u_n^{-1}\left(\hat{\theta}_{1,n} - \theta_0\right) \xrightarrow{\mathcal{L}_s} I(\theta_0)^{-1/2}\mathcal{N},$$

where $\mathcal{N}$ is a standard Gaussian variable independent of $I(\theta_0)$

Stable CIR process with Brownian component

We now consider

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dB_t + \delta X_{t-}^{1/\alpha} dL_t^{\alpha}, \quad t \geq 0, \quad x_0 > 0$$

- High-frequency observations : $(X_{\frac{i}{n}})_{i \in \{0,\dots,n\}}$
- Estimation of (σ, δ, α)

  $2a > \sigma^2 > 0, \quad \delta > 0, \quad \alpha \in (1, 2)$
- Problem : asymptotic bias due to the superposition of a Brownian Motion and a Lévy process with infinite variation

Toy model

To explain the problem, we consider the process

$$X_t = \sigma B_t + \delta L_t^\alpha, \quad \alpha \in (1,2)$$

Estimation of σ^2 and α from the increments $\Delta_j^n X = X_{\frac{j}{n}} - X_{\frac{j-1}{n}}$

Mancini (2009), Aït-Sahalia and Jacod (2009), Jacod and Todorov (2014-2016)
Bull (2016), Cooper Boniece, Figueroa-López and Han (2022)

- Truncated quadratic variation

$$\sum_{j=1}^{n} \Delta_j^n X^2 1_{|\Delta_j^n X| \leq \frac{u}{n^\tau}}, \quad u > 0, \quad 0 < \tau < \frac{1}{2}$$

- Real part of the characteristic function

$$L^n(u) = \frac{1}{n} \sum_{j=1}^{n} \cos(u\sqrt{n}\Delta_j^n X), \quad u > 0$$

- Number of large jumps

$$\sum_{j=1}^{n} 1_{|\Delta_j^n X| \geq \frac{u}{n^\tau}}, \quad u > 0, \quad 0 < \tau < \frac{1}{2}$$

Bias expansion

- Truncated quadratic variation

$$\mathbb{E}\Delta_j^n X^2 1_{|\Delta_j^n X|\le \frac{u}{n^\tau}} = \frac{\sigma^2}{n} + \frac{\sigma^2\delta^\alpha}{u^\alpha n^{2-\alpha\tau}} C_{1,\alpha} + \frac{\delta^\alpha u^{2-\alpha}}{n^{1+\tau(2-\alpha)}} C_{2,\alpha} + R_n$$

- Real part of the characteristic function

$$\mathbb{E}\cos(u\sqrt{n}\Delta_j^n X) = \mathcal{R}e^{-u^2\sigma^2/2} e^{-u^\alpha\delta^\alpha n^{\alpha/2-1}(1-i\tan(\pi\alpha/2))}$$

$$\mathbb{E}\cos(u\sqrt{n}(\Delta_j^n X - \Delta_{j+1}^n X)) = e^{-u^2\sigma^2} e^{-2u^\alpha\delta^\alpha n^{\alpha/2-1}} \quad \textit{symmetrised version}$$

- Number of large jumps

$$\mathbb{E}1_{|\Delta_j^n X|\ge \frac{u}{n^\tau}} = \frac{\delta^\alpha}{u^\alpha n^{1-\alpha\tau}} C_{1,\alpha} + \frac{\sigma^2\delta^\alpha}{u^{\alpha+2} n^{2-(\alpha+2)\tau}} C_{2,\alpha} + \frac{\delta^{2\alpha}}{u^{2\alpha} n^{2-2\alpha}} C_{3,\alpha} + R_n$$

Limit theorems and estimation

- Truncated quadratic variation

$$\sqrt{n}\left(\sum_{j=1}^{n}\Delta_j^n X^2 1_{|\Delta_j^n X|\le \frac{u}{n^\tau}} - \sigma^2 - \frac{\sigma^2\delta^\alpha}{u^\alpha n^{1-\alpha\tau}}C_{1,\alpha} - \frac{\delta^\alpha u^{2-\alpha}}{n^{\tau(2-\alpha)}}C_{2,\alpha}\right) \xrightarrow{\mathcal{L}_s} \mathcal{N}(0, 2\sigma^4)$$

Extension to a semimartingale : estimation of the integrated volatility (Cooper Boniece, Figueroa-López and Han (2022) use this result for different values of u to eliminate the bias, rate optimal and efficient with restrictions on α)

- Real part of the characteristic function (symmetrised version)

$$\sqrt{n}\left(L^n(u) - e^{-u^2\sigma^2}e^{-2u^\alpha\delta^\alpha n^{\alpha/2-1}}\right) \xrightarrow{\mathcal{L}_s} \mathcal{N}(0, \frac{e^{-4u^2\sigma^2} - 2e^{-2u^2\sigma^2} + 1}{2})$$

Extension to a semimartingale : estimation of the integrated volatility (Jacod and Todorov (2014-2016) local volatility estimation by taking the logarithm and combination with different values of u, rate optimal and efficient $\forall\alpha \in (0, 2)$)

- Number of large jumps

$$n^{-\alpha\tau}\sum_{j=1}^{n} 1_{|\Delta_j^n X|\ge \frac{u}{n^\tau}} \to \frac{\delta^\alpha}{u^\alpha}C_{1,\alpha}$$

Extension to a semimartingale : estimation of α (Aït-Sahalia and Jacod (2009) non optimal rate of estimation, Bull (2016) near optimal rate)

Stable CIR with Brownian component and infinite variation jump component

We come back to the process

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dB_t + \delta X_{t-}^{1/\alpha} dL_t^{\alpha}, \quad t \geq 0, \quad x_0 > 0$$

The previous methods work for the CIR process but they are not completely satisfactory.

We propose an alternative estimation method via estimating functions (*extension of Mies (2020)*)

- combining the real part of the characteristic function and smooth threshold exceeding
- based on the expansion of $\mathbb{E}f(u\sqrt{n}(\Delta_j^n X - \Delta_{j+1}^n X))$

We obtain the following results (forthcoming paper)

- joint estimation of $(\sigma^2, \delta, \alpha)$
- rate optimal and efficient estimator of σ^2 for $\alpha \in (1,2)$
- near rate optimal estimator of (δ, α)

Thank you for your attention

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