Functional convex ordering for stochastic processes: a constructive and simulable approach

Gilles Pagès
(includes collaborations with B. Jourdain, Y. Liu & C. Yeo)

LPSM-Sorbonne-Université

Rencontres Mathématiques de Rouen

Rouen 20 juin 2024

Definitions

Definition (Convex order, peacock)

(a) Two $\mathbb{R}^d$-valued random vectors $U, V \in L^1(\mathbb{P})$ are ordered w.r.t. convex order, denoted

$$U \preceq_{cvx} V$$

if, for every $\varphi : \mathbb{R}^d \to \mathbb{R}$, convex, [$\varphi$ Lipschitz is enough (Jourdain-P. 2023) by an inf convolution argument],

$$\mathbb{E}\,\varphi(U) \leq \mathbb{E}\,\varphi(V) \in (-\infty, +\infty].$$

(b) A stochastic process $(X_u)_{u\geq 0}$ is a p.c.o.c. (for "processus croissant pour l'ordre convexe") if

$u \longmapsto X_u$ is non-decreasing for the convex order.

- Then $\mathbb{E}\,U = \mathbb{E}\,V$ [$\varphi(x) = \pm x_i$, $i = 1:d$], and, if both lie in L^2 [$\varphi(x) = |x|^2$],

$$\mathrm{Var}(U) \leq \mathrm{Var}(V).$$

where $\mathrm{Var}(U) = \mathbb{E}\,|U - \mathbb{E}\,U|^2$.

Examples and motivation

- If $(X_t)_{t\geq 0}$ is a martingale, then $(X_t)_{t\geq 0}$ is a p.c.o.c./peacock: let $0 \leq s \leq t$,

$$\mathbb{E}\,\varphi(X_s) = \mathbb{E}\left(\varphi(\mathbb{E}(X_t|X_s))\right) \underbrace{\leq}_{Jensen} \mathbb{E}\left(\mathbb{E}(\varphi(X_t)|X_s)\right) = \mathbb{E}\,\varphi(X_t).$$

- Example: Gaussian distributions (centered): Let $Z \sim \mathcal{N}(0, I_q)$ on $\mathbb{R}^q$ and let $A,\ B \in \mathbb{M}(d, q)$ be $d \times q$ matrices

$$\big(A \preceq B \ \ i.e.\ \ BB^* - AA^* \in \mathcal{S}^+(d)\big) \Longleftrightarrow AZ \preceq_{cvx} BZ$$

i.e. $\mathcal{N}(0, AA^*) \preceq_{cvx} \mathcal{N}(0, BB^*)$ [Still true if Z is radial: $Z \sim OZ$, $\forall\, O \in O(d)$, Jourdain-P. 2022].

- Proof: Let $Z_1, Z_2 \sim \mathcal{N}(0; I_q)$ be independent and set

$$X_1 = AZ_1, \quad X_2 = X_1 + (BB^* - AA^*)^{1/2} Z_2.$$

Then (X_1, X_2) is an $\mathbb{R}^d$-valued martingale and $X_2 \sim \mathcal{N}(0, BB^*)$.

- Scalar case $d = q = 1$: $|\sigma| \leq |\vartheta| \Longleftrightarrow \mathcal{N}(0, \sigma^2) \preceq_{cvx} \mathcal{N}(0, \vartheta^2)$.
- 1D-proof: $\varphi : \mathbb{R} \to \mathbb{R}$ convex and $Z \in L^1$. Then, by Jensen's $\leq$,

$$u \mapsto \mathbb{E}\,\varphi(uZ) \text{ is convex and attains its minimum } \varphi(0) \text{ at } u = 0.$$

Hence $u \mapsto \mathbb{E}\,\varphi(uZ)$ is non-decreasing on $\mathbb{R}_+$ and non-increasing on $\mathbb{R}_-$.

About the converse of "martingale $\Rightarrow$ p.c.o.c."

- Strassen's Theorem (1965): $\mu \preceq_{cvx} \nu \Longleftrightarrow \exists$ transition $P(x, dy)$ s.t.

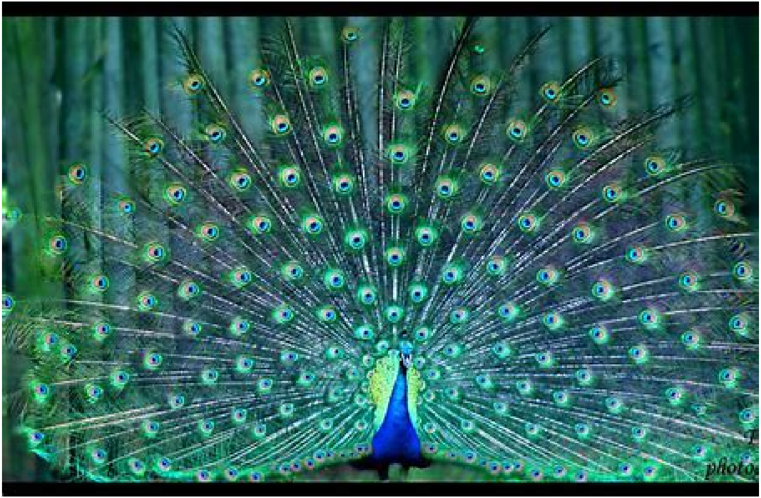
$$\nu = \mu P \quad \text{and} \quad \forall\, x \in \mathbb{R}^d, \quad \int y\, P(x, dy) = x.$$

- Kellerer's Theorem (1972): X is a p.c.o.c $\Longleftrightarrow$

  there exists a martingale $(M_t)_{t\geq 0}$ such that $X_t \stackrel{d}{=} M_t,\; t \geq 0$,

  i.e. X is a "1-martingale".
- Both proofs are unfortunately non-constructive.
- In Hirsch, Roynette, Profeta & Yor's monography, many (many...) explicit "representations" of p.c.o.c. by true martingales.

Representation of a p.c.o.c.



A revival motivated by Finance...

- A starter! t being fixed, $\sigma \mapsto e^{\sigma W_t - \frac{\sigma^2 t}{2}}$ is a p.c.o.c. since

$$\forall\, \sigma > 0, \quad e^{\sigma W_t - \frac{\sigma^2 t}{2}} \stackrel{d}{=} e^{W_{\sigma^2 t} - \frac{\sigma^2 t}{2}} \;(\to \sigma\text{-martingale}).$$

- Application to Black-Scholes model $S_t^\sigma = s_0 e^{\sigma W_t - \frac{\sigma^2 t}{2}}$. For every convex payoff function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$

$$0 \le \sigma \le \sigma' \Longrightarrow \mathbb{E}\,\varphi(S_t^\sigma) \le \mathbb{E}\,\varphi(S_t^{\sigma'}).$$

- Vanilla options: *Call* and *Put* options: $\varphi(S_T) = (S_T - K)^+$, $\varphi(S_T) = (K - S_T)^+$, etc.
- Path-dependent options (Asian payoffs). Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ convex

$$\sigma \longmapsto \mathrm{Premium}(\sigma) = \mathbb{E}\Big[\varphi\Big(\frac{1}{T}\int_0^T \underbrace{s_0 e^{\sigma W_t - \frac{\sigma^2 t}{2}}}_{=S_t^\sigma} dt\Big)\Big] \;?$$

- P. Carr et al. (2008): Non-decreasing in σ when $\varphi(x) = (x-K)^+$ (Asian Call) using PDEs.
- Baker-Yor (2010):

$$\sigma \mapsto \frac{1}{T}\int_0^T s_0 e^{\sigma W_t - \frac{\sigma^2 t}{2}} dt = \int_0^1 s_0 e^{W_{u\sigma^2 T} - \frac{u\sigma^2 T}{2}} du \stackrel{d}{=} \int_0^1 s_0 e^{W_{u,\sigma T} - \frac{u\sigma^2 T}{2}} du$$

where $(W_{u,t})_{u,t\geq 0}$ is a standard Brownian sheet ([1]). Hence a p.c.o.c. since

$$t \mapsto \int_0^1 s_0 e^{W_{u,t} - \frac{ut}{2}} du \text{ is an } (\mathcal{F}_{1,t})_{t\geq 0}\text{-martingale.}$$

with $\mathcal{F}_{1,t} = \sigma(W_{u,s},\, 0 \leq u \leq 1,\, 0 \leq s \leq t)$.

- Yields bounds on the option prices.

[1] $\mathbb{E}\, W_{u,t} W_{v,s} = (u \wedge v)(t \wedge s)$.

▷ This suggests many other (new or not so new) questions !

- Monotone (non-decreasing) convex order : $\exists$ drif b! [Hajek, 1985].
- Functional convex order I: switch from BS to local volatility models $i.e$ $\sigma \rightsquigarrow \sigma(x)$: $\sigma \mapsto \mathbb{E}\, f(X_T^{(\sigma)})$ [see e.g. El Karoui-Jeanblanc-Schreve, 1998].
- m-marginal path-dependent convex order: e.g. $\mathbb{E}\, f(X_{T_1}^{(\sigma)}, X_{T_2}^{(\sigma)})$ if $m = 2$. [see e.g.Brown, Rogers, Hobson 2001, Rüschendorf et al. 2008]
- Functional convex order II: from $\mathbb{E}\, f(X_T^{(\sigma)})$ to $\mathbb{E}\, F(X^{(\sigma)})$ i.e. path-dependent convex order [P.2016].
- Bermuda options [Pham 2005, Rüschendorf 2008], American options [P. 2016].
- Jump (risky asset) dynamics for $(X_t^{(\sigma)})$? [Rüschendorf-Bergenthum 2007, P. 2016]
- P.c.o.c. trough Martingale Optimal Transport. [Beïgelbock, Henry-Labordère et al., 2013, Tan et al. 2015, Jourdain-P. 2022].

Aims and methods

1. Unify and generalize these results with of focus on functional aspects (path-dependent payoffs) (like Asian options) i.e. both functional convex order I and II. With a...

2. Constraint: provide a constructive method of proof
   - based on time discretization of continuous time martingale dynamics (risky assets in Finance) .
   - using numerical schemes that preserve the functional convex order satisfied by the process under consideration...
   - e.g. to avoid "convexity arbitrages" in Finance.

3. Formulate a paradigm and apply it to various frameworks:
   - American style options, jump diffusions, stochastic integrals [P.2016, Sé. Prob.],
   - Stochastic Control and applications to swing options [P.-Yeo 2016],
   - McKean-Vlasov diffusions, MFG [Liu-P. 2022, SPA] and [Liu-P. 2023, AAP],
   - Volterra equations and rough volatility [Jourdain-P. Fin. & Stoch., to appear 2024],
   - Branching processes, Forward utilities, G-expectation ? Etc.

Example III: risk measure

- Let $X \in L^1\mathbb{P}$ be representative of a loss (with no atom for convenience) with c.d.f F_x.
- Let $\alpha \in (0,1]$, $\alpha \simeq 1$ be a risk level. Then

$$\mathrm{VaR}_\alpha(X) := (F_x)^{-1}(\alpha) \quad \text{and} \quad \mathrm{CVaR}_\alpha(X) := \mathbb{E}(X \,|\, X \geq \mathrm{Var}_\alpha(X))$$

- Rockafeller-Uryasev's representation of these two risk measures

$$L_{\alpha,X}(\xi) = \xi + \frac{1}{1-\alpha}\mathbb{E}\,(X-\xi)^+$$

satisfies $\partial_\xi L_{\alpha,X}(\xi) = 1 - \frac{1}{1-\alpha}\mathbb{P}(X \geq \xi)$.

$$\mathrm{VaR}_\alpha(X) = \mathrm{argmin}_{\mathbb{R}} L_{\alpha,X} \quad \text{and} \quad \mathrm{CVaR}_\alpha(X) = \min_{\mathbb{R}} L_{\alpha,X}.$$

- As a consequence

$$X \preceq_{icv} Y \Longrightarrow L_{\alpha,X} \leq L_{\alpha,Y}$$

so that

$$\mathrm{CVaR}_\alpha(X) \leq \mathrm{CVaR}_\alpha(Y).$$

- WARNING! Not true for the value-at-risk.

Martingale (and scaled) Brownian diffusions

- Pre-order $\preceq$ on $\mathcal{M}(d,q,\mathbb{R})$: let $A,\ B\in\mathbb{M}_{d,q}$.

$$A\preceq B \quad \text{if} \quad BB^*-AA^*\in\mathcal{S}^+(d,\mathbb{R}).$$

$\triangleright$ If $d=q=1$, $a\preceq b$ iff $a^2\le b^2$ iff $|a|\le|b|$

- $\preceq$-Convexity: $\sigma:\mathbb{R}^d\to\mathbb{M}_{d,q}$ is $\preceq$-convex if

$\forall\, x,\ y\in\mathbb{R}^d$, $\lambda\in[0,1]$, there exists $O_{\lambda,x},\ O_{\lambda,y}\in O(q)$ such that

$$\sigma\big(\lambda x+(1-\lambda)y\big)\preceq\lambda\sigma(x)O_{\lambda,x}+(1-\lambda)\sigma(y)O_{\lambda,y}$$

i.e.

$$\sigma\sigma^*\big(\lambda x+(1-\lambda)y\big)\le\big(\lambda\sigma(x)O_{\lambda,x}+(1-\lambda)\sigma(y)O_{\lambda,y}\big)\big(\lambda\sigma(x)O_{\lambda,x}+(1-\lambda)\sigma(y)O_{\lambda,y}\big)^*$$

- Criterion.

$$\sigma\sigma^*\big(\lambda x+(1-\lambda)y\big)\le\big(\lambda\sqrt{\sigma\sigma^*}(x)+(1-\lambda)\sqrt{\sigma\sigma^*}(y)\big)\big(\lambda\sqrt{\sigma\sigma^*}(x)+(1-\lambda)\sqrt{\sigma\sigma^*}(y)\big)^*$$

- $d=q=1$: $|\sigma|$ convex (with $O_{\lambda,x}=\mathrm{sign}(\sigma(x))$).

- $d,q\ge 2$: $\sigma(x)=A\,\mathrm{Diag}\big(\lambda_{1:q}(x)\big)O$, $A\in\mathbb{M}_{d,q}$, $O\in O(q)$ with

$$|\lambda_i|,\ i=1,\ldots,q,\ \text{convex}.$$

Theorem (martingale case (weak), P. 2016, Fadili-P. 2017, Jourdain-P. 2022)

Let $\sigma, \theta \in \mathcal{C}_{lin_x}\big([0,T]\times\mathbb{R}, \mathbb{M}_{d,q}\big)$.

$$dX_t^{(\sigma)} = \sigma(t, X_t^{(\sigma)})dW_t^{(\sigma)},\; X_0^{(\sigma)} \in L^{1+\eta},\; \eta > 0$$

$$dX_t^{(\theta)} = \theta(t, X_t^{(\theta)})dW_t^{(\theta)},\; X_0^{(\theta)} \in L^{1+\eta}, \quad \textit{both } (W_t^{(\cdot)})_{t\in[0,T]} \textit{ standard B.M.}$$

(a) *If* $X_0^{(\sigma)} \preceq_{cvx} X_0^{(\theta)}$ *and*

$$\begin{cases} (i)_\sigma & \sigma(t,.) : \mathbb{R}^d \to \mathbb{M}_{d,q} \text{ is } \preceq\text{-convex for every } t \in [0,T], \\ or & \\ (i)_\theta & \theta(t,.) : \mathbb{R}^d \to \mathbb{M}_{d,q} \text{ is } \preceq\text{-convex for every } t \in [0,T], \\ and & \\ (ii) & \sigma(t,\cdot) \preceq \theta(t,\cdot) \text{ for every } t \in [0,T] \end{cases}$$

then, for every functional l.s.c. convex fucnctional $F : \mathcal{C}([0,T], \mathbb{R}^d) \to \mathbb{R}$,

(i) *The function* $x \mapsto \mathbb{E}\, F(X^{(\sigma),x})$ *is convex from* $\mathbb{R}^d$ *to* $(-\infty, +\infty]$,

(ii) *Convex ordering holds:* $\qquad \mathbb{E}\, F(X^{(\sigma)}) \le \mathbb{E}\, F(X^{(\theta)}) \in (-\infty, +\infty]$.

• By a functional inf-convolution argument, it suffices to consider $\|\cdot\|_{\sup}$-Lipschitz functionals [Jourdain-P., Fin. & Stoch., 2024].

Theorem (martingale case (strong), P. 2016, Fadili-P. 2017, Jourdain-P. 2022)

Let $\sigma, \theta \in \mathrm{Lip}_x\big([0,T]\times\mathbb{R}, \mathbb{M}_{d,q}\big)$, *$W$ q-S.B.M.. Let $X^{(\sigma)}$ and $X^{(\theta)}$ be the* unique *strong solutions to*

$$dX_t^{(\sigma)} = \sigma(t, X_t^{(\sigma)})dW_t,\ X_0^{(\sigma)} \in L^1 \text{ (no more } \eta!),$$

$$dX_t^{(\theta)} = \theta(t, X_t^{(\theta)})dW_t,\ X_0^{(\theta)} \in L^1, \quad (W_t)_{t\in[0,T]} \text{ standard B.M.}$$

(a) *If* $X_0^{(\sigma)} \preceq_{cvx} X_0^{(\theta)}$ *and*

$$\begin{cases} (i)_\sigma & \sigma(t,.) : \mathbb{R}^d \to \mathbb{M}_{d,q} \text{ is } \preceq\text{-convex for every } t \in [0,T], \\ \text{or} & \\ (i)_\theta & \theta(t,.) : \mathbb{R}^d \to \mathbb{M}_{d,q} \text{ is } \preceq\text{-convex for every } t \in [0,T], \\ \text{and} & \\ (ii) & \sigma(t,\cdot) \preceq \theta(t,\cdot) \text{ for every } t \in [0,T] \end{cases}$$

then, for every l.s.c. convex functional $F : \mathcal{C}([0,T],\mathbb{R}^d) \to \mathbb{R}$,

(i) *The function* $x \mapsto \mathbb{E}\, F(X^{(\sigma),x})$ *is convex from* $\mathbb{R}^d$ *to* $(-\infty,+\infty]$,

(ii) *Convex ordering holds:* $\quad \mathbb{E}\, F(X^{(\sigma)}) \le \mathbb{E}\, F(X^{(\theta)}) \in (-\infty,+\infty]$.

- By a functional inf-convolution argument, it suffices to consider $\|\cdot\|_{\sup}$-Lipschitz functionals.

Scaled/drifted martingale diffusions (extension to)

- The former theorems still hold true for

$$dX_t^{(\sigma)} = \alpha(t)\big(X_t^{(\sigma)} + \beta(t)\big)dt + \sigma(t, X_t^{(\sigma)})dW_t^{(\sigma)},$$
$$dX_t^{(\theta)} = \alpha(t)\big(X_t^{(\theta)} + \beta(t)\big)dt + \theta(t, X_t^{(\theta)})dW_t^{(\theta)},$$

  where $\alpha(t) \in \mathbb{M}_{d,d}(\mathbb{R})$ and $\beta(t) \in \mathbb{R}^d$ are continuous.
- Change of variable:

$$\widetilde{X}_t^{(\sigma)} = e^{-\int_0^t \alpha(s)ds}\big(X_t^{(\sigma)} + \beta(t)\big).$$

- Finance: spot interest rate $\alpha(t) = r(t)\mathbf{1}$ and $\beta(t) = 0$ since typical (risk-neutral) dynamics of traded assets read

$$dS_t = r(t)S_t dt + S_t\sigma(S_t, \omega)dW_t$$

- For more general drifts $b(t, x)$ when $d = q = 1$: functional version of Hajek's theorem: monotone functional convex order holds true if

$$\forall\, t \in [0, T], \quad b(t, .) \text{ is convex.}$$

Strategy of proof (constructive)

- Time discretization (preferably) accessible to simulation: typically the Euler scheme.
- Propagate convexity (marginal or pathwise)
- Propagate comparison (marginal or pathwise)
- Transfer by functional limit theorems "à la Jacod-Shiryaev" (weak setting) or L^1 $\|\cdot\|_{\sup}$-convergence of the "regular" Euler scheme.

Step 1: discrete time ARCH models

- ARCH dynamics: Let $(Z_k)_{1\leq k\leq n}$ be a sequence of independent, symmetric r.v. on $(\Omega, \mathcal{A}, \mathbb{P})$. Two ARCH models: $X_0, Y_0 \in L^1(\mathbb{P})$,

$$\begin{aligned} X_{k+1} &= X_k + \sigma_k(X_k)\, Z_{k+1}, \\ Y_{k+1} &= Y_k + \theta_k(Y_k)\, Z_{k+1}, \quad k = 0 : n-1, \end{aligned}$$

where σ_k, $\theta_k : \mathbb{R} \to \mathbb{R}$, $k = 0 : n-1$ have linear growth.

Proposition (Propagation result)

If σ_k, $k = 0 : n-1$ are $\preceq$-convex with linear growth,

$$X_0 = x \quad \text{and} \quad \forall\, k \in \{0, \ldots, n-1\}, \quad \sigma_k \preceq \theta_k,$$

then, for every convex funtion $F : (\mathbb{R}^d)^{n+1} \to \mathbb{R}$ convex with linear growth

$$x \longmapsto \mathbb{E}\, F(x, X_1^x \ldots, X_n^x) \quad \text{is convex.}$$

Partial proof (marginal) with Gaussian white noise

- $Z_k \stackrel{d}{=} \mathcal{N}(0, I_q)$, $1 \le k \le n$, or simply radial.
- Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function. Let

$$P_k^\sigma f(x) := \mathbb{E} f(x + \sigma_k(x)Z) = \big[\mathbb{E}\, f(x + AZ)\big]_{|A=\sigma_k(x)}.$$

- Set $\mathbb{M}_{d,q} \ni A \mapsto Qf(A) := \mathbb{E}\, f(x + AZ)$ is right $O(q)$-invariant, convex and $\preceq$-non-decreasing for convex ordering by the starting example.
- Then $P_k^\sigma f$ is convex since $\forall\, x,\, y \in \mathbb{R}^d$ and $\forall\, \lambda \in [0, 1]$

$$\begin{aligned} P_k^\sigma f\big(\lambda x + (1-\lambda)y\big) &= Qf\big(\sigma_k(\lambda x + (1-\lambda)y)\big) \\ &\le Qf\big(\lambda \sigma_k(x) + (1-\lambda)\sigma_k(y)\big) \\ &\le \lambda Qf\big(\sigma_k(x)\big) + (1-\lambda) Qf\big(\sigma_k(y)\big) \\ &= \lambda P_k^\sigma f(x) + (1-\lambda) P_k^\sigma f(y). \end{aligned}$$

- Hence

$$x \longmapsto \mathbb{E}\, f(X_n^x) = P_{1:n}^\sigma f(x) := P_1^\sigma \circ \cdots \circ P_n^\sigma f(x) \quad \text{is convex}$$

Theorem (Discrete time comparison result)

If all σ_k, $k = 0 : n-1$ or all θ_k, $k = 0 : n-1$ are $\preceq$-convex with linear growth,

$$X_0 \preceq_{cvx} Y_0 \quad \text{and} \quad \forall\, k \in \{0, \ldots, n-1\}, \quad \sigma_k \preceq \theta_k,$$

then

$$(X_0, \ldots, X_n) \preceq_{cvx} (Y_0, \ldots, Y_n).$$

Partial proof (1-marginal) with Gaussian white noise

- Backward induction on k.
- For $k = n$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a (Lipschitz) convex function.

$$\sigma_n \preceq \theta_n \Longrightarrow P_n^{\sigma} f(x) = Qf\big(\sigma_n(x)\big) \le Qf\big(\theta_n(x)\big) = P_n^{\theta} f(x)$$

by non-decreasing $\preceq$-monotony of Q.

- Assume $\underbrace{P^{\sigma}_{k+1:n} f}_{\text{convex}} := P^{\sigma}_{k+1} \circ \cdots \circ P^{\sigma}_n \le P^{\theta}_{k+1:n} f$. Then

$$A \in \mathbb{M}_{d,q} \longmapsto Q\big(P^{\sigma}_{k+1:n} f\big)(A) \quad \text{is } \preceq\text{-non-decreasing}$$

so that

$$P^{\sigma}_{k:n} f(x) = P^{\sigma}_k (P^{\sigma}_{k+1:n}) f(x) = Q\big(P^{\sigma}_{k+1:n} f\big)\big(\sigma_k(x)\big) \overset{\downarrow}{\le} Q\big(P^{\sigma}_{k+1:n} f\big)\big(\theta_k(x)\big)$$

$$Q \text{ is a positive operator} \le Q\big(P^{\theta}_{k+1:n} f\big)\big(\theta_k(x)\big)$$

$$= P^{\theta}_{k:n} f(x).$$

- Hence

$$\mathbb{E}\, f(X_n^{\sigma}) = \mathbb{E}\, P^{\sigma}_{1:n} f(X_0) \le \mathbb{E}\, P^{\sigma}_{1:n} f(Y_0) \le \mathbb{E}\, P^{\theta}_{1:n} f(Y_0) = \mathbb{E}\, f(X_n^{\theta}).$$

Functional approach

- By "functional" we mean here : $F(X_0, \ldots, X_n)$ with $F : (\mathbb{R}^d)^{n+1} \to \mathbb{R}$ convex.
- Same strategy by induction
- But entirely backward.

Step 2 of the proof: Back to continuous time

▷ Euler scheme(s): Discrete time Euler scheme with step $\frac{T}{n}$, starting at x is an ARCH model. For $X^{(\sigma)}$: for $k = 0, \ldots, n-1$,

$$\bar{X}^{(\sigma),n}_{t^n_{k+1}} = \bar{X}^{(\sigma),n}_{t^n_k} + \sigma(t^n_k, \bar{X}^{(\sigma),n}_{t^n_k})\big(W_{t^n_{k+1}} - W_{t^n_k}\big), \; \bar{X}^{(\sigma),n}_0 = x$$

Set

$$Z_k = W_{t^n_k} - W_{t^n_{k-1}}, \; k = 1, \ldots, n$$

$$\Downarrow$$

discrete time setting applies

Remark. Linear growth of σ and θ, implies that if $X^{(\sigma)}_0, X^{(\theta)}_0 \in L^p(\mathbb{P})$ for some $p = 1 + \eta > 1$, then

$$\sup_{n\geq 1}\Big\| \sup_{t\in[0,T]} |\bar{X}^{(\sigma),n}_t|\Big\|_p + \sup_{n\geq 1}\Big\| \sup_{t\in[0,T]} |\bar{X}^{(\theta),n}_t|\Big\|_p \leq C(1+\|X^{(\sigma)}_0\|_p + \|X^{(\theta)}_0\|_p) < +\infty.$$

From discrete to continuous time

▷ Interpolation ($n \geq 1$)

- *Piecewise affine interpolator* defined by

$$\forall\, x_{0:n} \in \mathbb{R}^{n+1},\ \forall\, k = 0, \ldots, n-1,\ \forall\, t \in [t_k^n, t_{k+1}^n], \quad .$$

$$i_n(x_{0:n})(t) = \frac{n}{T}\Big((t_{k+1}^n - t)x_k + (t - t_k^n)x_{k+1}\Big)$$

- $\widetilde{X}^{(\sigma),n} := i_n\Big((\bar{X}^{(\sigma),n}_{t_k^n})_{k=0:n}\Big) =$ piecewise affine Euler scheme.

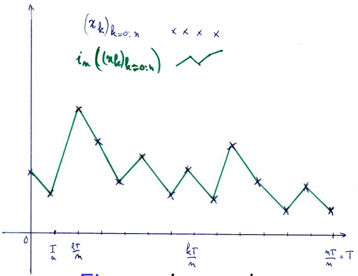


Figura: *Interpolator*

▷ Let $F : \mathcal{C}([0,T],\mathbb{R}) \to \mathbb{R}$ be a convex functional (with r-poly. growth).

$$\forall\, n \geq 1, \qquad F_n : \mathbb{R}^{n+1} \ni x_{0:n} \longmapsto F_n(x_{0:n}) := F\big(i_n(x_{0:n})\big).$$

- Step 1 (Discrete time): $F\big(\widetilde{X}^{(\sigma),n}\big) = F_n\big((\bar{X}^{(\sigma),n}_{t^n_k})_{k=0:n}\big)$ and

$$F \text{ convex} \Longrightarrow F_n \text{ convex}, \ n \geq 1.$$

Discrete time result implies since $\sigma(t^n_k,.) \leq \theta(t^n_k,.)$.

$$\mathbb{E}\, F\big(\widetilde{X}^{(\sigma),n}\big) = \mathbb{E}\, F_n\big((\bar{X}^{(\sigma),n}_{t^n_k})_{k=0:n}\big) \leq \mathbb{E}\, F_n\big((\bar{X}^{(\theta),n}_{t^n_k})_{k=0:n}\big) = \mathbb{E}\, F\big(\widetilde{X}^{(\theta),n}\big).$$

- Step 2 (Weak transfer here): See e.g. [Jacod-Shiryaev's book, 2^{nd} edition, Theorem 3.39, p.551].

$$\widetilde{X}^{(\sigma),n} \stackrel{\mathcal{L}(\|.\|_{\sup})}{\longrightarrow} X^{(\sigma)} \qquad \text{as } n \to \infty.$$

combined with uniform integrability (by $L^{1+\eta}(\mathbb{P})$-boundedness) yield

$$\mathbb{E}\, F(X^{(\sigma)}) = \lim_n \mathbb{E}\, F\big(\widetilde{X}^{(\sigma),n}\big) \qquad (\textit{idem} \text{ for } X^{(\theta)}).$$

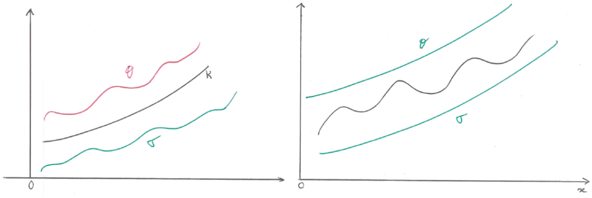


Figura: *How to use this result ?*

The Euler scheme provides a simulable approximation

which preserves convex order.

Is convexity necessary ? $\sigma(t,x) = \sigma(x)$, $d = 1$

- Note that when $\vartheta = \sigma$, *a posteriori* ordering $\Rightarrow$ convexity since

$$\delta_{\lambda x+(1-\lambda)y} \preceq_{cvx} \lambda\delta_x + (1-\lambda)\delta_y$$

  and $\sigma(\cdot) \preceq \sigma(\cdot)$ (sic!) so that

$$\mathbb{E}\, F(X^{\lambda x+(1-\lambda)y}) \leq \lambda\mathbb{E}\, F(X^x) + (1-\lambda)\mathbb{E}\, F(X^y).$$

- One shows [Jourdain-P '23] that (when $d = 1$)

$$\sqrt{\frac{2}{\pi}}|\sigma(x)| = \lim_{t\to 0}\frac{1}{\sqrt{t}}\mathbb{E}|X_t^x - x| = \lim_{t\to 0}\frac{1}{\sqrt{t}}\mathbb{E}|X_t^x - X_0^x| = \lim_{t\to 0}\frac{1}{\sqrt{t}}\mathbb{E}F(X^x)$$

  with $F(\alpha) = |\alpha(t) - \alpha(0)|$ an (only) 2-marginal functional convex functional.
- As sooon as convexity propagation for 2-marginal functional holds true then $|\sigma|$ is convex !!
- The convexity assumption on σ or ϑ is mandatory ...except maybe for 1-marginal convex order when $d = q = 1$.

For the $1D$ diffusion (after [El Karoui et al.])

- Let $\varphi(x) = \mathbb{E}\, f(X_T^x)$ with $f : \mathbb{R} \to \mathbb{R}$ convex with right derivative f'_r.
- One has by pathwise differentiation

$$\begin{aligned}\varphi'(x) &= \mathbb{E}\Big[f'_r(X_T^x)e^{\int_0^T \sigma'(X_s^x)dW_s - \frac{1}{2}\int_0^T (\sigma')^2(X_s^x)ds}\Big]\\ &= \dots\\ &= \mathbb{E}_{\mathbb{Q}} f'_r(Y_T^x) \quad \text{Girsanov !}\end{aligned}$$

- $x \mapsto Y_T^x$ is non-decreasing (under light assumptions on σ, cf. e.g. [Revuz-Yor])
- Finally, f'_r being non-decreasing,

$$\varphi' : x \longmapsto \mathbb{E}_{\mathbb{Q}}\, f'_r(X_T^x) \text{ is non-decreasing}$$

so that (almost ...) whatever σ is

$$\varphi : x \longmapsto \mathbb{E}\, f(X_T^x) \text{ is convex.}$$

- So $1D$ setting for 1-marginal functionals is special !

Conclusions (for diffusions)

- True functional convex ordering needs convexity of σ or θ in any dimension.
- Convexity of true functionals $x \mapsto \mathbb{E}\,F(X^{\sigma,x})$ needs convexity of σ in any dimension.
- In one dimension, 1-marginal convexity propagation and ordering does not need convexity: it's fro free.
- And that's it (for brownian diffusions) !

- If you are (really) interested to know more (1D jumpy Lévy driven SDE, Optimal stopping and American options, Stochastic Control and swing options/gas storage, McKean-Vlasov SDEs, Volterra equations, (not Markov !) etc), have a look at my 2024 Bachelier course. Slides available at

www.bachelier-paris.fr/cours/

Merci de votre attention !